

Black hole radiation of massive spin-2 particles in (3+1) dimensions

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Abstract. This paper is devoted to the study of radiation of a massive spin-2 boson (graviton with a nonzero mass) through the event horizon of a generic static and spherically symmetric black hole in (3+1) dimensions. To this end, we consider the problem in the framework of the quantum tunneling phenomenon. We evaluate the tunneling rate of the massive gravitons by applying the semiclassical WKB approximation to the Fierz-Pauli equation. The temperature of the radiation is obtained with the aid of the Boltzmann expression. Our findings are in good agreement with the existing Hawking radiation studies in the current literature.

1 Introduction

Hawking [1–4] theoretically proved the existence of black hole radiation: a thermal (black-body) emission. It is named after him as the Hawking radiation (HR). This radiation originates in fact from the steady conversion of quantum vacuum fluctuations into pairs of particles, one of which is escaping towards spatial infinity while the other is caught by the black hole at the event horizon. This radiation reduces the mass of black holes and is therefore also known as black hole evaporation. Furthermore, each evaporation of different black holes possesses a characteristic temperature which is the so-called Hawking temperature. Since Hawking's seminal papers, many physicists have been studying the different derivations of the HR for the numerous spacetime models including black holes, wormholes, black strings, etc., by using different kinds of quantum particles (spin-0, 1/2, 1, 3/2 ...). For example, a reader may refer to [5–40].

One of the most well-known methods in the computation of the HR is to use the WKB approximation [41] in the complex path integration method of Srinivasan *et al.* [8, 12]. In this approach, one usually considers energetic quantum particles which penetrate the black hole through its horizon. Such quanta have classically forbidden (+) / acceptable (–) trajectories whose action (S_{\pm}) is suitable for the WKB approximation method. The differential equation of the action can be obtained by substituting the proper wave function, $\Phi \sim e^{\frac{i}{\hbar}S(t,r,\theta,\phi)}$, into the wave equation of a considered curved spacetime. Then, the Hamilton-Jacobi method [42] is employed to solve the differential equation for the action. In general, the most contributive terms of this method are found in the leading order of the Planck constant \hbar [13–16]. The emission and absorption probabilities of the tunneling particles are given by [12]

$$\Gamma_{em} = \exp\left(-\frac{2ImS_+}{\hbar}\right), \quad (1)$$

$$\Gamma_{ab} = \exp\left(-\frac{2ImS_-}{\hbar}\right), \quad (2)$$

such that their ratio yields the tunneling probability as follows:

$$\Gamma = \frac{\Gamma_{em}}{\Gamma_{ab}} = e^{-\beta E}, \quad (3)$$

where β denotes the inverse temperature and E is the energy [17]. Since the problem is nothing but the finding of the ratio (3), we can always follow a way which normalizes the absorption probability to 1 (%100 engulfment)

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thereof the classical description of the black hole. At the same time, this classical condition on Γ_{ab} enables us to read the quantum mechanical output: the emission probability. Thus, the tunneling probability becomes identical to the emission probability: $\Gamma = \Gamma_{em}$. Similar to this methodology, there exist other ways to obtain Γ (see, for instance, refs. [36, 37]).

A first important step toward the understanding of elementary and composite particles in a relativistic context was made by Klein [43] and Gordon [44]. Their resultant equation, the so-called Klein-Gordon equation, is one of the powerful equations that describe the dynamics of the relativistic spin-0 particles (scalar bosons, pions, scalar mesons, and also Higgs particles) in any geometry. However, in nature, many elementary particles have spin and Dirac [45] introduced the correct form of the equation for fermions with spin-1/2 (such as electrons, quarks, and leptons). Then, Proca [46] derived an equation for the massive spin-1 particles (vector particles: \mathcal{W} and \mathcal{Z} bosons and gluons) which play a crucial role in the fundamental interactions as being the force carriers: the \mathcal{W} and \mathcal{Z} bosons for the weak interaction, and the gluons for the strong interaction. The Proca equation reduces to the Maxwell equation with the limit of the zero masses (photons). In the standard model, experimental verification of the fundamental particles has only been made for the particles having spin 1 and less. However, theoretical physics prescribes that a graviton (massless boson), which is the mediator of the gravitational force, must have spin-2.

Nowadays, one of the trend subjects in physics is the massive gravity theory [47–49]. This is a theory of gravity that modifies general relativity by endowing the graviton with mass. In the classical theory, this means that gravitational waves obey a massive wave equation and hence travel at speeds below the speed of light [50]. A theory of a massive spin-2 field propagating in a spacetime background was discovered long ago by Fierz and Pauli [51, 52]: the so-called Fierz-Pauli equations (FPEs). However, it was understood in the 1970s that the massive graviton theories have some drawbacks. These theories mainly suffer from the inevitable inclusion of ghost modes and the problem of the general relativity limit while the graviton's mass vanishes. Although some solutions to those problems exist in three dimensions [53–55], they could not be solved in four dimensions until the work of D'Amico *et al.* [56]. On the other hand, LIGO and Virgo Collaborations' observations have recently confirmed the existence of gravitational waves [57]. Although these experiments cannot detect individual gravitons, they might provide information about certain properties of the graviton.

Very recently, FPEs have been reformulated by Koenigstein *et al.* [58], which is going to be employed in our present study for the derivation of the HR arising from quantum tunneling of the massive gravitons through the event horizon of a generic static and spherically symmetric (3+1)-dimensional black hole. To the best of our knowledge, this problem has not been studied before. In this respect, the present study aims to fill this gap in the literature.

2 Quantum tunneling of massive spin-2 particles

In this section, we will focus on the quantum tunneling of the massive spin-2 bosons from a generic static and spherically symmetric (3+1)-dimensional black hole. To this end, we consider the metric as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2[d\theta^2 + (\sin(\theta))^2 d\phi^2]. \quad (4)$$

The horizon of such a black hole (4) is obtained via the condition of $f(r_h) = 0$.

According to the recent study of Koenigstein *et al.* [58] (see its eq. (44)), the FPEs are given by

$$\mathcal{G}_{\mu\nu} = \square T_{\mu\nu} - \nabla_\mu \nabla_\nu T + \frac{m^2}{\hbar^2} T_{\mu\nu} - \frac{m^2}{\hbar^2} g_{\mu\nu} T = 0, \quad (5)$$

where ∇_μ denotes the covariant derivative, \square is the Laplacian operator,

$$\square = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \partial^\nu), \quad (6)$$

and $T_{\mu\nu}$ represents the spin-2 fields (symmetric rank-2 tensor) which can be expressed as [59]

$$T_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu - \frac{1}{2} g_{\mu\nu} \nabla^\beta \Phi_\beta, \quad (7)$$

in which $\Phi_\nu = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ stands for the two-component spinor field. To obtain approximate solutions to the FPEs (5), we use the following the WKB ansatz for the spinor field

$$\Phi_\nu = (c_0, c_1, c_2, c_3) e^{\frac{i}{\hbar} S(t, r, \theta, \phi)}, \quad (8)$$

where (c_0, c_1, c_2, c_3) represent arbitrary constants. The action seen in eq. (8) is given by [23]

$$S(t, r, \theta, \phi) = S_0(t, r, \theta, \phi) + \hbar S_1(t, r, \theta, \phi) + \hbar^2 S_2(t, r, \theta, \phi) + \dots \quad (9)$$

After substituting eqs. (8) and (9) in eq. (5), we obtain sixteen equations which can be seen in the appendix. However, due to the symmetry eqs. (A.1)–(A.10) reduce a tetrad equation set that allows us to read the elements (to the lowest order in \hbar) of the coefficient matrix $\Lambda(c_0, c_1, c_2, c_3)^T = 0$, where the superscript T denotes the transposition, as follows:

$$\begin{aligned} \Lambda_{11} = & i(f(r))^3 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial r} S_0(t, r, \theta, \phi) + ir^2(\sin(\theta))^2(f(r))^4 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^3 \\ & + i(f(r))^3(\sin(\theta))^2 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^2 - i \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right) r^2 m^2 (f(r))^3 (\sin(\theta))^2, \end{aligned} \quad (10)$$

$$\begin{aligned} \Lambda_{12} = & ir^2(\sin(\theta))^2 \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right)^3 - if(r)(\sin(\theta))^2 \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^2 \\ & + i \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) r^2 m^2 f(r) (\sin(\theta))^2 - if(r) \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial t} S_0(t, r, \theta, \phi), \end{aligned} \quad (11)$$

$$\Lambda_{13} = \Lambda_{14} = ir^2(\sin(\theta))^2(f(r))^2 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi), \quad (12)$$

$$\begin{aligned} \Lambda_{21} = & ir^4(\sin(\theta))^4(f(r))^3 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) + ir^2(\sin(\theta))^2(f(r))^2 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^2 \\ & \times \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) + ir^2(\sin(\theta))^4(f(r))^2 \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^3 - i \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right) r^4 (\sin(\theta))^4 m^2 (f(r))^2, \end{aligned} \quad (13)$$

$$\Lambda_{22} = \Lambda_{24} = ir^4(\sin(\theta))^4 f(r) \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi), \quad (14)$$

$$\begin{aligned} \Lambda_{23} = & ir^6(\sin(\theta))^4 \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right)^3 - ir^4(\sin(\theta))^2 f(r) \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \\ & - ir^6(\sin(\theta))^4 (f(r))^2 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial t} S_0(t, r, \theta, \phi) + i \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) r^6 (\sin(\theta))^4 m^2 f(r), \end{aligned} \quad (15)$$

$$\begin{aligned} \Lambda_{31} = & ir^2 \sin(\theta) (f(r))^2 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^3 + ir^4 (\sin(\theta))^3 (f(r))^3 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2 \\ & + ir^2 (\sin(\theta))^3 (f(r))^2 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^2 - i \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) r^4 (\sin(\theta))^3 m^2 (f(r))^2, \end{aligned} \quad (16)$$

$$\Lambda_{32} = \Lambda_{33} = ir^4 (\sin(\theta))^3 f(r) \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right) \frac{\partial}{\partial t} S_0(t, r, \theta, \phi), \quad (17)$$

$$\begin{aligned} \Lambda_{34} = & ir^6 (\sin(\theta))^5 \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right)^3 - ir^6 (\sin(\theta))^5 (f(r))^2 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \\ & + i \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) r^6 (\sin(\theta))^5 m^2 f(r) - ir^4 (\sin(\theta))^5 f(r) \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^2, \end{aligned} \quad (18)$$

$$\Lambda_{41} = \Lambda_{42} = ir^4 (\sin(\theta))^3 f(r) \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right) \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi), \quad (19)$$

$$\begin{aligned} \Lambda_{43} = & ir^6 (\sin(\theta))^3 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right)^2 - ir^4 f(r) \sin(\theta) \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right)^3 \\ & + i \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) r^6 f(r) m^2 (\sin(\theta))^3 - ir^6 (f(r))^2 (\sin(\theta))^3 \left(\frac{\partial}{\partial \phi} S_0(t, r, \theta, \phi) \right) \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2, \end{aligned} \quad (20)$$

$$\begin{aligned} \Lambda_{44} = & i \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right) r^6 (\sin(\theta))^5 m^2 f(r) + ir^6 (\sin(\theta))^5 \left(\frac{\partial}{\partial t} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \\ & - ir^4 (\sin(\theta))^5 f(r) \left(\frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi) \right)^3 - ir^6 (\sin(\theta))^5 (f(r))^2 \left(\frac{\partial}{\partial r} S_0(t, r, \theta, \phi) \right)^2 \frac{\partial}{\partial \theta} S_0(t, r, \theta, \phi). \end{aligned} \quad (21)$$

Having non-trivial solutions is conditional on $\det A = 0$ [23, 29]. Using the following Hamilton-Jacobi solution for the lowest-order action,

$$S_0(t, r, \theta, \phi) = -Et + W(r) + J\phi + K(\theta) + \Omega, \tag{22}$$

where E and J are the energy and angular momentum of the spin-2 particle, respectively, and Ω is a complex constant [23], the equation of $\det A = 0$ becomes

$$\begin{aligned} & \left(f(r)(\sin(\theta))^2 \left(\frac{d}{d\theta} K(\theta) \right)^2 + r^2(\sin(\theta))^2 \left(\frac{d}{dr} W(r) \right)^2 (f(r))^2 - r^2(E^2 + f(r)m^2)(\sin(\theta))^2 + f(r)J^2 \right)^3 \\ & \times \left(-f(r)(\sin(\theta))^2 \left(\frac{d}{d\theta} K(\theta) \right)^2 - r^2(\sin(\theta))^2 \left(\frac{d}{dr} W(r) \right)^2 (f(r))^2 + r^2(E^2 + 2f(r)m^2)(\sin(\theta))^2 f(r)J^2 \right) \\ & \times r^{10}(\sin(\theta))^6 E^2 = 0. \end{aligned} \tag{23}$$

From above, we obtain the following integral solutions for $W(r)$:

$$W_{\pm}(r) = \pm \int \frac{\sqrt{\left(-J^2 - (\sin(\theta))^2 \left(\frac{d}{d\theta} K(\theta)\right)^2 + r^2 m^2 (\sin(\theta))^2\right) f(r) + r^2 (\sin(\theta))^2 E^2}}{r \sin(\theta) f(r)} \tag{24}$$

and

$$W_{\pm}(r) = \pm \int \frac{\sqrt{\left(-J^2 - (\sin(\theta))^2 \left(\frac{d}{d\theta} K(\theta)\right)^2 + 2r^2 m^2 (\sin(\theta))^2\right) f(r) + r^2 (\sin(\theta))^2 E^2}}{r \sin(\theta) f(r)}, \tag{25}$$

where $+$ ($-$) corresponds to the outgoing (ingoing) spin-2 particle. Here, one can ask what would happen if we took into account the van Dam-Veltman-Zakharov (vDVZ) discontinuity [60, 61] (the limit of vanishing mass of the spin-2 particle) in the above integral solutions. Since we shall consider eqs. (24) and (25) around the horizon, one can easily see that the expressions (including the mass (m) terms) which are the factors of the metric function $f(r)$ in the square roots are eliminated at the horizon: $f(r_h) = 0$. Namely, the zero-mass limit is smooth during the quantum tunneling phenomenon. The latter result is in agreement with the conclusion of the previous studies [62–64]: “the vDVZ discontinuity is only present for a flat background metric”.

On the other hand, each $W_{\pm}(r)$ would have a simple pole at the horizon. This pole problem can be overcome by employing the complex path integration method prescribed in [12]. To this end, we first expand the metric function $f(r)$ around the horizon:

$$f(r) \approx \left. \frac{d}{dr} f(r) \right|_{r=r_h} + \mathcal{O}(r - r_h)^2. \tag{26}$$

Then, we insert eq. (26) into integral solutions (24) and (25), and subsequently evaluate the integrals. Thus, it can be seen that eqs. (24) and (25) give the same answer:

$$W_{\pm}(r) \approx \frac{\pm \pi i E}{\left. \frac{d}{dr} f(r) \right|_{r=r_h}}. \tag{27}$$

Hence, we infer from eq. (22) that imaginary parts of the action come from both eq. (27) and the imaginary part of the complex constant Ω . So, the emission and absorption probabilities of the spin-2 particles crossing the event horizon each way can be expressed as [13]

$$\Gamma_{em} = \exp\left(-\frac{2\text{Im}S_+}{\hbar}\right) = \exp\left[-\frac{2}{\hbar}(\text{Im}W_+ + \text{Im}\Omega)\right], \tag{28}$$

$$\Gamma_{ab} = \exp\left(-\frac{2\text{Im}S_-}{\hbar}\right) = \exp\left[-\frac{2}{\hbar}(\text{Im}W_- + \text{Im}\Omega)\right]. \tag{29}$$

According to the classical definition of the black hole, everything is swallowed by it. This condition can be fulfilled by simply normalizing the absorption probability to unity: $\Gamma_{ab} = 1$. Thus, we see that $\text{Im}\Omega = -\text{Im}W_-$. On the other

hand, since $W_- = -W_+$, the latter condition is nothing short of $\text{Im } \Omega = \text{Im } W_+$, and it then yields the tunneling rate

$$\begin{aligned} \Gamma &= \frac{\Gamma_{em}}{\Gamma_{ab}} = \Gamma_{em}, \\ &= \exp\left(-\frac{4}{\hbar} \text{Im } W_+\right), \\ &= \exp\left(\frac{-4\pi E}{\left.\frac{d}{dr}f(r)\right|_{r=r_h}}\right). \end{aligned} \quad (30)$$

Comparing eq. (30) with eq. (3), we read the radiation temperature which is associated with the event horizon of the (3+1)-dimensional BH as follows:

$$T = \frac{\hbar}{4\pi} \left.\frac{d}{dr}f(r)\right|_{r=r_h}. \quad (31)$$

The above temperature is indeed the original Hawking temperature of the static and symmetric black hole which is verified many times by the quantum tunnelings of the other particles having spin other than two [13–15,26]. Namely, the radiation temperature obtained in eq. (31) is the reproduction of the standard Hawking temperature of the metric (4).

3 Conclusion

In this paper, we have studied the quantum tunneling of spin-2 massive particles (bosons) from a generic (3+1)-dimensional static and symmetric black hole. By applying the WKB approximation (to the leading order in \hbar) and using the proper Hamilton-Jacobi ansatz in the FPEs (see eqs. (A.1)–(A.10)), we have managed to obtain the quantum tunneling rate (30) of the massive spin-2 particles emitted from that generic black hole. The corresponding black-body (from the Boltzmann formula) temperature (31) of the radiation matches the well-known Hawking temperature of the metric (4), which is recurrently verified by other particles' quantum tunnelings. Meanwhile, during the quantum tunneling computations, it has been seen that the vDVZ discontinuity does not play a role in the obtained tunneling rate (30).

It is worth noting that one can also compute the non-thermal radiation to the radiation temperature by taking into account higher orders in \hbar . However, such a study, which is in our agenda, requires more advanced computations compared to the present paper, and its results are expected to be remarkable from the aspect of information loss paradox [7]. Moreover, the generalization of the present result to the other spacetimes like higher-dimensional black holes, rotating black holes, dynamic black holes, black strings, and wormholes may reveal more information than this work. Those will also be our next problems in the near future.

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Appendix A.

The sixteen FPEs (5) (due to the symmetry ($\mathcal{G}_{\alpha\beta} = \mathcal{G}_{\beta\alpha}$)) are as follows:

$$\begin{aligned} \mathcal{G}_{tt} &= \left[\left(\frac{\partial^3}{\partial t^3} \Phi_0 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 + 2 \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{\partial^2}{\partial t^2} \Phi_1 \right) \frac{d}{dr} f(r) - \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 \right]^2 \\ &\times \left(\frac{\partial}{\partial t} \Phi_0 \right) f(r) - 4 \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial^2}{\partial t \partial r} \Phi_0 + \hbar^2 r^2 (\sin(\theta))^2 \Phi_1 \left(\frac{d}{dr} f(r) \right)^3 \\ &- \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 \left(\frac{\partial}{\partial r} \Phi_1 \right) f(r) - 2 \hbar^2 (f(r))^3 r^2 (\sin(\theta))^2 \frac{\partial^3}{\partial t \partial r^2} \Phi_0 - 2 \hbar^2 (f(r))^2 \\ &\times (\sin(\theta))^2 \frac{\partial^3}{\partial \theta^2 \partial t} \Phi_0 - 4 \hbar^2 (f(r))^3 (\sin(\theta))^2 r \frac{\partial^2}{\partial t \partial r} \Phi_0 - 2 \hbar^2 (f(r))^2 \frac{\partial^3}{\partial t \partial \phi^2} \Phi_0 - 2 \hbar^2 (f(r))^2 \\ &\times \sin(\theta) \cos(\theta) \frac{\partial^2}{\partial \theta \partial t} \Phi_0 - \left(\frac{\partial^3}{\partial t^2 \partial r} \Phi_1 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 - \left(\frac{\partial^3}{\partial \theta \partial t^2} \Phi_2 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{\partial^3}{\partial t^2 \partial \phi} \Phi_3 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 - m^2 (f(r))^2 \left(\frac{\partial}{\partial t} \Phi_0 \right) r^2 (\sin(\theta))^2 + m^2 (f(r))^2 r^2 (\sin(\theta))^2 \\
& \times \left[\frac{\partial}{\partial r} \Phi_1 + m^2 (f(r))^2 r^2 (\sin(\theta))^2 \frac{\partial}{\partial \theta} \Phi_2 + m^2 (f(r))^2 r^2 (\sin(\theta))^2 \frac{\partial}{\partial \phi} \Phi_3 \right] \Big/ [r^2 (\sin(\theta))^2 f(r) \hbar^2], \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\theta t} = \mathcal{G}_{t\theta} = & \left[-2 \hbar^2 (\sin(\theta))^3 r^2 \left(\frac{\partial}{\partial t} \Phi_2 \right) (f(r))^2 - 2 \hbar^2 (\cos(\theta))^2 (f(r))^2 \left(\frac{\partial}{\partial \theta} \Phi_0 \right) \sin(\theta) - 2 m^2 r^4 \right. \\
& \times (\sin(\theta))^3 f(r) \frac{\partial}{\partial t} \Phi_2 + 2 m^2 r^2 (\sin(\theta))^3 (f(r))^2 \frac{\partial}{\partial \theta} \Phi_0 - 2 \hbar^2 (\sin(\theta))^3 (f(r))^3 \frac{\partial}{\partial \theta} \Phi_0 \\
& - 4 \hbar^2 \cos(\theta) (f(r))^2 \frac{\partial^2}{\partial \phi^2} \Phi_0 + 2 \hbar^2 (f(r))^2 \left(\frac{\partial^3}{\partial \theta \partial \phi^2} \Phi_0 \right) \sin(\theta) + 2 \hbar^2 (\sin(\theta))^3 \\
& (f(r))^2 \frac{\partial^3}{\partial \theta^3} \Phi_0 + 2 \hbar^2 r^4 (\sin(\theta))^3 \frac{\partial^3}{\partial t^3} \Phi_2 + \hbar^2 (\sin(\theta))^3 (f(r))^2 r^2 \left(\frac{d^2}{dr^2} f(r) \right) \frac{\partial}{\partial \theta} \Phi_0 \\
& + \hbar^2 (\sin(\theta))^3 f(r) r^4 \left(\frac{d^2}{dr^2} f(r) \right) \frac{\partial}{\partial t} \Phi_2 + 4 \hbar^2 r^2 (\sin(\theta))^3 \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial^2}{\partial \theta \partial r} \Phi_0 \\
& - 2 \hbar^2 f(r) r^2 \left(\frac{\partial^2}{\partial \theta \partial t} \Phi_2 \right) (\sin(\theta))^2 \cos(\theta) + 4 \hbar^2 f(r) r^2 (\sin(\theta))^2 \left(\frac{\partial^2}{\partial t \partial \phi} \Phi_3 \right) \cos(\theta) \\
& - 2 \hbar^2 r^4 (\sin(\theta))^3 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial^2}{\partial t \partial r} \Phi_2 \right) f(r) - 4 \hbar^2 (\sin(\theta))^3 f(r) r \frac{\partial^2}{\partial \theta \partial t} \Phi_1 - 2 \hbar^2 (\sin(\theta))^3 \\
& \times (f(r))^2 r^4 \frac{\partial^3}{\partial t \partial r^2} \Phi_2 + 2 \hbar^2 (\sin(\theta))^3 (f(r))^3 r^2 \frac{\partial^3}{\partial \theta \partial r^2} \Phi_0 + 4 \hbar^2 (\sin(\theta))^3 (f(r))^3 \left(\frac{\partial^2}{\partial \theta \partial r} \Phi_0 \right) r \\
& - 2 \hbar^2 r^2 (\sin(\theta))^3 \left(\frac{\partial^2}{\partial \theta \partial t} \Phi_1 \right) \frac{d}{dr} f(r) - 8 \hbar^2 (\sin(\theta))^3 (f(r))^2 r^3 \frac{\partial^2}{\partial t \partial r} \Phi_2 + 2 \left(\frac{\partial^3}{\partial \theta \partial t \partial \phi} \Phi_3 \right) \\
& \times r^2 (\sin(\theta))^3 f(r) \hbar^2 + 2 \left(\frac{\partial^3}{\partial \theta \partial t \partial r} \Phi_1 \right) r^2 (\sin(\theta))^3 f(r) \hbar^2 - 2 \hbar^2 f(r) r^2 \left(\frac{\partial^3}{\partial t \partial \phi^2} \Phi_2 \right) \sin(\theta) \\
& + 2 \hbar^2 (\sin(\theta))^2 \cos(\theta) (f(r))^2 \frac{\partial^2}{\partial \theta^2} \Phi_0 - 2 \hbar^2 r^3 (\sin(\theta))^3 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial t} \Phi_2 \right) f(r) \\
& \left. + 2 \hbar^2 r (\sin(\theta))^3 \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial}{\partial \theta} \Phi_0 + 2 \hbar^2 f(r) (\cos(\theta))^2 r^2 \left(\frac{\partial}{\partial t} \Phi_2 \right) \sin(\theta) \right] \Big/ [-2 r^2 (\sin(\theta))^3 f(r) \hbar^2], \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\phi t} = \mathcal{G}_{t\phi} = & \left[2 \hbar^2 (f(r))^2 \left(\frac{\partial}{\partial \phi} \Phi_0 \right) (\sin(\theta))^2 - 2 \hbar^2 (\sin(\theta))^2 (f(r))^3 \frac{\partial}{\partial \phi} \Phi_0 + 2 \hbar^2 (\sin(\theta))^2 (f(r))^2 \right. \\
& \times \frac{\partial^3}{\partial \theta^2 \partial \phi} \Phi_0 + 2 \hbar^2 r^4 (\sin(\theta))^4 \frac{\partial^3}{\partial t^3} \Phi_3 + 2 \hbar^2 r^2 (\sin(\theta))^4 \left(\frac{\partial}{\partial t} \Phi_3 \right) f(r) - 2 \hbar^2 r^2 (\sin(\theta))^4 \\
& \times \left(\frac{\partial}{\partial t} \Phi_3 \right) (f(r))^2 - 2 m^2 r^4 (\sin(\theta))^4 f(r) \frac{\partial}{\partial t} \Phi_3 + 2 m^2 r^2 (\sin(\theta))^2 (f(r))^2 \frac{\partial}{\partial \phi} \Phi_0 \\
& + 2 \left(\frac{\partial^3}{\partial t \partial r \partial \phi} \Phi_1 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 + 4 \hbar^2 (\sin(\theta))^2 (f(r))^3 \left(\frac{\partial^2}{\partial r \partial \phi} \Phi_0 \right) r + 2 \hbar^2 (\sin(\theta))^2 \\
& \times (f(r))^3 r^2 \frac{\partial^3}{\partial r^2 \partial \phi} \Phi_0 - 2 \hbar^2 (\sin(\theta))^4 (f(r))^2 r^4 \frac{\partial^3}{\partial t \partial r^2} \Phi_3 - 8 \hbar^2 (\sin(\theta))^4 (f(r))^2 r^3 \\
& \times \frac{\partial^2}{\partial t \partial r} \Phi_3 - 2 \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{\partial^2}{\partial t \partial \phi} \Phi_1 \right) \frac{d}{dr} f(r) + 2 \left(\frac{\partial^3}{\partial \theta \partial t \partial \phi} \Phi_2 \right) r^2 (\sin(\theta))^2 f(r) \hbar^2 \\
& - 2 \hbar^2 f(r) (\sin(\theta))^4 r^2 \frac{\partial^3}{\partial \theta^2 \partial t} \Phi_3 - 4 \hbar^2 f(r) r (\sin(\theta))^2 \frac{\partial^2}{\partial t \partial \phi} \Phi_1 + 2 \hbar^2 \cos(\theta) (f(r))^2 \\
& \times \left(\frac{\partial^2}{\partial \theta \partial \phi} \Phi_0 \right) \sin(\theta) + 2 \hbar^2 (f(r))^2 \frac{\partial^3}{\partial \phi^3} \Phi_0 - 2 \hbar^2 r^3 (\sin(\theta))^4 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial t} \Phi_3 \right) f(r) \\
& \left. + 2 \hbar^2 r (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial}{\partial \phi} \Phi_0 + \hbar^2 (\sin(\theta))^4 f(r) r^4 \left(\frac{d^2}{dr^2} f(r) \right) \frac{\partial}{\partial t} \Phi_3 \right]
\end{aligned}$$

$$\begin{aligned}
 & + \hbar^2(\sin(\theta))^2(f(r))^2r^2 \left(\frac{d^2}{dr^2}f(r) \right) \frac{\partial}{\partial\phi}\Phi_0 - 6\hbar^2f(r)(\sin(\theta))^3\cos(\theta)r^2 \\
 & \times \frac{\partial^2}{\partial\theta\partial t}\Phi_3 + 4\hbar^2r^2(\sin(\theta))^2 \left(\frac{d}{dr}f(r) \right) (f(r))^2 \frac{\partial^2}{\partial r\partial\phi}\Phi_0 - 4\hbar^2f(r)\sin(\theta)\cos(\theta)r^2 \\
 & \times \frac{\partial^2}{\partial t\partial\phi}\Phi_2 - 2\hbar^2r^4(\sin(\theta))^4 \left(\frac{d}{dr}f(r) \right) \left(\frac{\partial^2}{\partial t\partial r}\Phi_3 \right) f(r) \Big] / [-2r^2(\sin(\theta))^2f(r)\hbar^2], \tag{A.3}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{r\theta} = \mathcal{G}_{\theta r} = & \left[10\hbar^2(\sin(\theta))^3(f(r))^2 \frac{\partial}{\partial\theta}\Phi_1 + 2\hbar^2r^2(\sin(\theta))^3 \frac{\partial^3}{\partial\theta\partial t^2}\Phi_1 + 4\hbar^2f(r)\cos(\theta) \right. \\
 & \times \frac{\partial^2}{\partial\phi^2}\Phi_1 - 2\hbar^2f(r)\sin(\theta) \frac{\partial^3}{\partial\theta\partial\phi^2}\Phi_1 - 2\hbar^2(\sin(\theta))^3f(r) \frac{\partial^3}{\partial\theta^3}\Phi_1 \\
 & + 2 \left(\frac{\partial^3}{\partial\theta\partial r\partial\phi}\Phi_3 \right) (f(r))^2r^2(\sin(\theta))^3\hbar^2 + 4\hbar^2r^3(\sin(\theta))^3 \left(\frac{\partial^2}{\partial t^2}\Phi_2 \right) f(r) + 2\hbar^2r^4(\sin(\theta))^3 \\
 & \times \left(\frac{\partial^3}{\partial t^2\partial r}\Phi_2 \right) f(r) - 2\hbar^2r^4(\sin(\theta))^3 \left(\frac{d}{dr}f(r) \right) \frac{\partial^2}{\partial t^2}\Phi_2 + 20\hbar^2(\sin(\theta))^3(f(r))^3r\Phi_2 \\
 & - 2\hbar^2(\sin(\theta))^3(f(r))^3r^2 \frac{\partial}{\partial r}\Phi_2 + 2\hbar^2f(r)\sin(\theta)(\cos(\theta))^2 \frac{\partial}{\partial\theta}\Phi_1 - 4m^2r^3(\sin(\theta))^3(f(r))^2\Phi_2 \\
 & - 2m^2r^4(\sin(\theta))^3(f(r))^2 \frac{\partial}{\partial r}\Phi_2 - 2m^2r^2(\sin(\theta))^3f(r) \frac{\partial}{\partial\theta}\Phi_1 - 2\hbar^2\sin(\theta)r^2 \\
 & \times \left(\frac{\partial^3}{\partial r\partial\phi^2}\Phi_2 \right) (f(r))^2 + 4\hbar^2(\sin(\theta))^3(f(r))^2r \frac{\partial^2}{\partial\theta^2}\Phi_2 + 4\hbar^2(f(r))^2(\sin(\theta))^3r \frac{\partial^2}{\partial\theta\partial\phi}\Phi_2 \\
 & - 2\hbar^2f(r)(\sin(\theta))^2\cos(\theta) \frac{\partial^2}{\partial\theta^2}\Phi_1 - 8\hbar^2(\sin(\theta))^3(f(r))^2r \frac{\partial^2}{\partial\theta\partial r}\Phi_1 - 2\hbar^2(\sin(\theta))^3(f(r))^3 \\
 & \times r^4 \frac{\partial^3}{\partial r^3}\Phi_2 - 12\hbar^2(\sin(\theta))^3(f(r))^3r^3 \frac{\partial^2}{\partial r^2}\Phi_2 + 2 \left(\frac{\partial^3}{\partial\theta\partial t\partial r}\Phi_0 \right) (f(r))^2r^2(\sin(\theta))^3\hbar^2 \\
 & - 14\hbar^2r^3(\sin(\theta))^3 \left(\frac{d}{dr}f(r) \right) (f(r))^2 \frac{\partial}{\partial r}\Phi_2 + 10\hbar^2r(\sin(\theta))^3 \left(\frac{d}{dr}f(r) \right) f(r) \frac{\partial}{\partial\theta}\Phi_1 \\
 & - 4\hbar^2(\sin(\theta))^3(f(r))^2r^2\Phi_2 \frac{d}{dr}f(r) + 4\hbar^2\sin(\theta)(\cos(\theta))^2r\Phi_2(f(r))^2 + 4\hbar^2(\sin(\theta))^2 \\
 & \times \cos(\theta)r \left(\frac{\partial}{\partial\theta}\Phi_2 \right) (f(r))^2 + 2\hbar^2\sin(\theta)(\cos(\theta))^2r^2 \left(\frac{\partial}{\partial r}\Phi_2 \right) (f(r))^2 + 8\hbar^2r(\sin(\theta))^2 \\
 & \times \left(\frac{\partial}{\partial\phi}\Phi_3 \right) (f(r))^2\cos(\theta) - \hbar^2(\sin(\theta))^3(f(r))^2r^4 \left(\frac{\partial}{\partial r}\Phi_2 \right) \frac{d^2}{dr^2}f(r) \\
 & + \hbar^2(\sin(\theta))^3f(r)r^2 \left(\frac{d^2}{dr^2}f(r) \right) \frac{\partial}{\partial\theta}\Phi_1 - 4\hbar^2r^4(\sin(\theta))^3 \left(\frac{d}{dr}f(r) \right) (f(r))^2 \frac{\partial^2}{\partial r^2}\Phi_2 \\
 & - 2\hbar^2(\sin(\theta))^3(f(r))^2r^3\Phi_2 \frac{d^2}{dr^2}f(r) + 4\hbar^2r^2(\sin(\theta))^2 \left(\frac{\partial^2}{\partial r\partial\phi}\Phi_3 \right) (f(r))^2\cos(\theta) \\
 & - 2\hbar^2(\sin(\theta))^2\cos(\theta)r^2 \left(\frac{\partial^2}{\partial\theta\partial r}\Phi_2 \right) (f(r))^2 + 2\hbar^2r^2(\sin(\theta))^3 \left(\frac{d}{dr}f(r) \right) \left(\frac{\partial^2}{\partial\theta\partial t}\Phi_0 \right) \\
 & \left. \times f(r) \right] / [-2\hbar^2(\sin(\theta))^3(f(r))^2r^2], \tag{A.4}
 \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\theta\theta} = & \left[2\hbar^2 r^3 (\sin(\theta))^3 \frac{\partial^3}{\partial\theta\partial t^2} \Phi_2 - 2\hbar^2 r^3 (\sin(\theta))^3 \left(\frac{\partial^2}{\partial\theta\partial r} \Phi_2 \right) f(r) \frac{d}{dr} f(r) - 2\hbar^2 (f(r))^2 r^3 (\sin(\theta))^3 \right. \\
& \frac{\partial^3}{\partial\theta\partial r^2} \Phi_2 - \left(\frac{\partial^3}{\partial\theta^3} \Phi_2 \right) r (\sin(\theta))^3 f(r) \hbar^2 - 4\hbar^2 (\sin(\theta))^3 (f(r))^2 r \frac{\partial}{\partial\theta} \Phi_2 \\
& - 8\hbar^2 (\sin(\theta))^3 (f(r))^2 r^2 \frac{\partial^2}{\partial\theta\partial r} \Phi_2 - 4\hbar^2 (\sin(\theta))^3 f(r) \frac{\partial^2}{\partial\theta^2} \Phi_1 + 4\hbar^2 (\sin(\theta))^3 f(r) r \Phi_1 \\
& \times \frac{d}{dr} f(r) - 4\hbar^2 (\sin(\theta))^3 (f(r))^2 r \frac{\partial}{\partial r} \Phi_1 - 2\hbar^2 f(r) r \left(\frac{\partial^3}{\partial\theta\partial\phi^2} \Phi_2 \right) \sin(\theta) + 4\hbar^2 f(r) r \sin(\theta) \\
& \times \left(\frac{\partial}{\partial\phi} \Phi_3 \right) (\cos(\theta))^2 + 4\hbar^2 f(r) r \cos(\theta) (\sin(\theta))^2 \frac{\partial^2}{\partial\theta\partial\phi} \Phi_3 + 4\hbar^2 f(r) r \cos(\theta) \\
& \times \frac{\partial^2}{\partial\phi^2} \Phi_2 - 2\hbar^2 f(r) r \left(\frac{\partial^2}{\partial\theta^2} \Phi_2 \right) (\sin(\theta))^2 \cos(\theta) + 4\hbar^2 f(r) r \left(\frac{\partial}{\partial\theta} \Phi_2 \right) (\cos(\theta))^2 \sin(\theta) \\
& + \left(\frac{\partial^3}{\partial\theta^2\partial t} \Phi_0 \right) r (\sin(\theta))^3 f(r) \hbar^2 + \left(\frac{\partial^3}{\partial\theta^2\partial r} \Phi_1 \right) r (\sin(\theta))^3 f(r) \hbar^2 + \left(\frac{\partial^3}{\partial\theta^2\partial\phi} \Phi_3 \right) \\
& \times r (\sin(\theta))^3 f(r) \hbar^2 - m^2 r^3 \left(\frac{\partial}{\partial\theta} \Phi_2 \right) (\sin(\theta))^3 f(r) + m^2 r^3 (\sin(\theta))^3 f(r) \frac{\partial}{\partial t} \Phi_0 + m^2 r^3 (\sin(\theta))^3 \\
& \left. f(r) \frac{\partial}{\partial r} \Phi_1 + m^2 r^3 (\sin(\theta))^3 f(r) \frac{\partial}{\partial\phi} \Phi_3 \right] / \left[-\hbar^2 (\sin(\theta))^3 f(r) r \right], \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\phi\phi} = & \left[4\hbar^2 (\sin(\theta))^3 f(r) r \Phi_1 \frac{d}{dr} f(r) - 4\hbar^2 f(r) r \sin(\theta) \left(\frac{\partial}{\partial\phi} \Phi_3 \right) (\cos(\theta))^2 - 6\hbar^2 f(r) r \cos(\theta) (\sin(\theta))^2 \right. \\
& \times \frac{\partial^2}{\partial\theta\partial\phi} \Phi_3 - 4\hbar^2 f(r) r \left(\frac{\partial}{\partial\theta} \Phi_2 \right) (\cos(\theta))^2 \sin(\theta) - 8\hbar^2 r (\sin(\theta))^2 \Phi_2 (f(r))^2 \cos(\theta) \\
& - 4\hbar^2 r^2 (\sin(\theta))^2 \left(\frac{\partial}{\partial r} \Phi_2 \right) (f(r))^2 \cos(\theta) - 2 \left(\frac{\partial^3}{\partial\theta^2\partial\phi} \Phi_3 \right) r (\sin(\theta))^3 f(r) \hbar^2 \\
& - 4\hbar^2 (\sin(\theta))^3 (f(r))^2 r \frac{\partial}{\partial r} \Phi_1 + \hbar^2 f(r) r \left(\frac{\partial^3}{\partial\theta\partial\phi^2} \Phi_2 \right) \sin(\theta) \\
& - 4\hbar^2 f(r) r \cos(\theta) \frac{\partial^2}{\partial\phi^2} \Phi_2 + m^2 r^3 (\sin(\theta))^3 f(r) \frac{\partial}{\partial t} \Phi_0 + m^2 r^3 (\sin(\theta))^3 f(r) \frac{\partial}{\partial r} \Phi_1 \\
& - m^2 r^3 (\sin(\theta))^3 f(r) \frac{\partial}{\partial\phi} \Phi_3 + 2\hbar^2 r^3 (\sin(\theta))^3 \frac{\partial^3}{\partial t^2\partial\phi} \Phi_3 \\
& - 4\hbar^2 f(r) \sin(\theta) \frac{\partial^2}{\partial\phi^2} \Phi_1 + m^2 r^3 \left(\frac{\partial}{\partial\theta} \Phi_2 \right) (\sin(\theta))^3 f(r) \\
& + \left(\frac{\partial^3}{\partial t\partial\phi^2} \Phi_0 \right) f(r) r \sin(\theta) \hbar^2 + \left(\frac{\partial^3}{\partial r\partial\phi^2} \Phi_1 \right) f(r) r \sin(\theta) \hbar^2 \\
& - \left(\frac{\partial^3}{\partial\phi^3} \Phi_3 \right) f(r) r \sin(\theta) \hbar^2 - 2\hbar^2 (f(r))^2 r^3 (\sin(\theta))^3 \frac{\partial^3}{\partial r^2\partial\phi} \Phi_3 - 8\hbar^2 r^2 (\sin(\theta))^3 (f(r))^2 \frac{\partial^2}{\partial r\partial\phi} \Phi_3 \\
& - 4\hbar^2 r (\sin(\theta))^3 \left(\frac{\partial}{\partial\phi} \Phi_3 \right) (f(r))^2 - 4\hbar^2 f(r) (\sin(\theta))^2 \left(\frac{\partial}{\partial\theta} \Phi_1 \right) \cos(\theta) \\
& \left. - 2\hbar^2 r^3 (\sin(\theta))^3 \left(\frac{\partial^2}{\partial r\partial\phi} \Phi_3 \right) f(r) \frac{d}{dr} f(r) \right] / \left[-\hbar^2 f(r) \sin(\theta) r \right], \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{rr} = & \left[2 \hbar^2 f(r) \left(\frac{d}{dr} f(r) \right) r \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \Phi_1 - 4 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) f(r) \Phi_1 \frac{d^2}{dr^2} f(r) \right. \\
 & + \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 \left(\frac{\partial}{\partial t} \Phi_0 \right) f(r) - \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 \left(\frac{\partial}{\partial r} \Phi_1 \right) f(r) \\
 & - 4 \hbar^2 f(r) (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 r^2 \Phi_1 + 4 \hbar^2 (f(r))^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) r^2 \frac{\partial}{\partial r} \Phi_1 \\
 & - 8 \hbar^2 (f(r))^2 (\sin(\theta))^2 r \Phi_1 \frac{d}{dr} f(r) + 8 \hbar^2 (f(r))^3 r \sin(\theta) \cos(\theta) \Phi_2 + 4 \hbar^2 (f(r))^3 \\
 & \times r^2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial r} \Phi_2 + 2 m^2 f(r) r^3 (\sin(\theta))^2 \Phi_1 \frac{d}{dr} f(r) + 2 \hbar^2 r^3 (\sin(\theta))^2 \\
 & \times \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial^2}{\partial t \partial r} \Phi_0 + 4 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d^2}{dr^2} f(r) \right) \left(\frac{\partial}{\partial r} \Phi_1 \right) (f(r))^2 \\
 & + 2 \hbar^2 f(r) (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) r \frac{\partial^2}{\partial \theta^2} \Phi_1 + 2 \hbar^2 r^3 (\sin(\theta))^2 (f(r))^2 \Phi_1 \frac{d^3}{dr^3} f(r) \\
 & + 4 \hbar^2 (f(r))^2 (\sin(\theta))^2 r^2 \Phi_1 \frac{d^2}{dr^2} f(r) - 2 \hbar^2 (f(r))^2 r \sin(\theta) \cos(\theta) \frac{\partial^2}{\partial \theta \partial r} \Phi_1 \\
 & + 4 \hbar^2 (f(r))^2 (\sin(\theta))^2 \frac{\partial^2}{\partial \theta^2} \Phi_1 - 2 \hbar^2 (f(r))^2 r \frac{\partial^3}{\partial r \partial \phi^2} \Phi_1 + 4 \hbar^2 (f(r))^3 \\
 & \times (\sin(\theta))^2 r \frac{\partial}{\partial \theta} \Phi_2 + 8 \hbar^2 (f(r))^3 (\sin(\theta))^2 r \frac{\partial}{\partial r} \Phi_1 \\
 & + 4 \hbar^2 (f(r))^3 r (\sin(\theta))^2 \frac{\partial}{\partial \phi} \Phi_3 + 4 \hbar^2 (f(r))^2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \Phi_1 \\
 & - \left(\frac{\partial^3}{\partial r^3} \Phi_1 \right) (f(r))^3 r^3 (\sin(\theta))^2 \hbar^2 + \left(\frac{\partial^3}{\partial t \partial r^2} \Phi_0 \right) (f(r))^3 r^3 (\sin(\theta))^2 \hbar^2 \\
 & + \left(\frac{\partial^3}{\partial r^2 \partial \phi} \Phi_3 \right) (f(r))^3 r^3 (\sin(\theta))^2 \hbar^2 - 4 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{\partial^2}{\partial t^2} \Phi_1 \right) \frac{d}{dr} f(r) \\
 & + 4 \hbar^2 (f(r))^3 r^2 (\sin(\theta))^2 \frac{\partial^2}{\partial r \partial \phi} \Phi_3 + 2 \hbar^2 f(r) \left(\frac{d}{dr} f(r) \right) r \frac{\partial^2}{\partial \phi^2} \Phi_1 \\
 & - 4 \hbar^2 (f(r))^3 (\sin(\theta))^2 r^2 \frac{\partial^2}{\partial r^2} \Phi_1 + 4 \hbar^2 (f(r))^3 (\sin(\theta))^2 r^2 \frac{\partial^2}{\partial \theta \partial r} \Phi_2 \\
 & - 2 \hbar^2 (f(r))^2 (\sin(\theta))^2 r \frac{\partial^3}{\partial \theta^2 \partial r} \Phi_1 + 2 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{\partial^3}{\partial t^2 \partial r} \Phi_1 \right) f(r) \\
 & + \hbar^2 r^3 (\sin(\theta))^2 \Phi_1 \left(\frac{d}{dr} f(r) \right)^3 - m^2 (f(r))^2 r^3 (\sin(\theta))^2 \frac{\partial}{\partial r} \Phi_1 \\
 & + m^2 (f(r))^2 r^3 (\sin(\theta))^2 \frac{\partial}{\partial t} \Phi_0 + m^2 (f(r))^2 r^3 (\sin(\theta))^2 \frac{\partial}{\partial \theta} \Phi_2 \\
 & \left. + m^2 (f(r))^2 r^3 (\sin(\theta))^2 \frac{\partial}{\partial \phi} \Phi_3 + 4 \hbar^2 (f(r))^2 \frac{\partial^2}{\partial \phi^2} \Phi_1 \right] / \left[-(f(r))^3 r^3 (\sin(\theta))^2 \hbar^2 \right], \tag{A.7}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{\phi r} = \mathcal{G}_{r\phi} = & \left[2 \hbar^2 (\sin(\theta))^4 (f(r))^2 r^2 \frac{\partial}{\partial r} \Phi_3 - 4 \hbar^2 (f(r))^2 (\sin(\theta))^4 r \Phi_3 - 2 \hbar^2 (\sin(\theta))^4 (f(r))^3 r^2 \frac{\partial}{\partial r} \Phi_3 \right. \\
 & + 20 \hbar^2 (f(r))^3 (\sin(\theta))^4 r \Phi_3 - 4 m^2 r^3 (\sin(\theta))^4 (f(r))^2 \Phi_3 - 2 m^2 r^4 (\sin(\theta))^4 (f(r))^2 \frac{\partial}{\partial r} \Phi_3 \\
 & - 2 m^2 r^2 (\sin(\theta))^2 f(r) \frac{\partial}{\partial \phi} \Phi_1 - 2 \hbar^2 (\sin(\theta))^4 r^2 (f(r))^2 \frac{\partial^3}{\partial \theta^2 \partial r} \Phi_3 + 2 \hbar^2 r^4 (\sin(\theta))^4 \\
 & \times \left(\frac{\partial^3}{\partial t^2 \partial r} \Phi_3 \right) f(r) - 2 \hbar^2 r^4 (\sin(\theta))^4 \left(\frac{d}{dr} f(r) \right) \frac{\partial^2}{\partial t^2} \Phi_3 \\
 & \left. - 12 \hbar^2 (\sin(\theta))^4 (f(r))^3 r^3 \frac{\partial^2}{\partial r^2} \Phi_3 + 4 \hbar^2 r^3 (\sin(\theta))^4 \left(\frac{\partial^2}{\partial t^2} \Phi_3 \right) f(r) \right]
 \end{aligned}$$

$$\begin{aligned}
& + 2 \left(\frac{\partial^3}{\partial t \partial r \partial \phi} \Phi_0 \right) (f(r))^2 r^2 (\sin(\theta))^2 \hbar^2 + 2 \left(\frac{\partial^3}{\partial \theta \partial r \partial \phi} \Phi_2 \right) (f(r))^2 r^2 (\sin(\theta))^2 \hbar^2 \\
& - 2 \hbar^2 f(r) \cos(\theta) \sin(\theta) \frac{\partial^2}{\partial \theta \partial \phi} \Phi_1 + 4 \hbar^2 (f(r))^2 r \left(\frac{\partial^2}{\partial \theta \partial \phi} \Phi_2 \right) (\sin(\theta))^2 \\
& + 4 \hbar^2 (f(r))^2 (\sin(\theta))^2 r \frac{\partial^2}{\partial \phi^2} \Phi_3 - 8 \hbar^2 (\sin(\theta))^2 (f(r))^2 \frac{\partial^2}{\partial r \partial \phi} \Phi_1 r \\
& - 2 \hbar^2 (\sin(\theta))^4 (f(r))^3 r^4 \frac{\partial^3}{\partial r^3} \Phi_3 - 2 \hbar^2 f(r) \frac{\partial^3}{\partial \phi^3} \Phi_1 - 14 \hbar^2 r^3 (\sin(\theta))^4 \\
& \times \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial}{\partial r} \Phi_3 + 10 \hbar^2 r (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) f(r) \frac{\partial}{\partial \phi} \Phi_1 + 8 \hbar^2 (f(r))^2 r (\sin(\theta))^3 \left(\frac{\partial}{\partial \theta} \Phi_3 \right) \\
& \times \cos(\theta) - 4 \hbar^2 \sin(\theta) (f(r))^2 r \cos(\theta) \frac{\partial}{\partial \phi} \Phi_2 - 4 \hbar^2 (\sin(\theta))^4 (f(r))^2 r^2 \Phi_3 \frac{d}{dr} f(r) + 16 \hbar^2 r (\sin(\theta))^2 \Phi_3 \\
& \times (f(r))^2 (\cos(\theta))^2 - \hbar^2 (\sin(\theta))^4 (f(r))^2 r^4 \left(\frac{\partial}{\partial r} \Phi_3 \right) \frac{d^2}{dr^2} f(r) + \hbar^2 (\sin(\theta))^2 f(r) r^2 \left(\frac{d^2}{dr^2} f(r) \right) \frac{\partial}{\partial \phi} \Phi_1 \\
& - 4 \hbar^2 r^4 (\sin(\theta))^4 \left(\frac{d}{dr} f(r) \right) (f(r))^2 \frac{\partial^2}{\partial r^2} \Phi_3 - 2 \hbar^2 (\sin(\theta))^4 (f(r))^2 r^3 \Phi_3 \frac{d^2}{dr^2} f(r) - 6 \hbar^2 (\sin(\theta))^3 r^2 \\
& \times (f(r))^2 \left(\frac{\partial^2}{\partial \theta \partial r} \Phi_3 \right) \cos(\theta) - 4 \hbar^2 \sin(\theta) r^2 \cos(\theta) (f(r))^2 \frac{\partial^2}{\partial r \partial \phi} \Phi_2 + 2 \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \\
& \times \left(\frac{\partial^2}{\partial t \partial \phi} \Phi_0 \right) f(r) - 2 \hbar^2 (\sin(\theta))^2 f(r) \frac{\partial}{\partial \phi} \Phi_1 + 10 \hbar^2 (\sin(\theta))^2 (f(r))^2 \frac{\partial}{\partial \phi} \Phi_1 \\
& - 2 \hbar^2 f(r) (\sin(\theta))^2 \frac{\partial^3}{\partial \theta^2 \partial \phi} \Phi_1 + 2 \hbar^2 r^2 (\sin(\theta))^2 \frac{\partial^3}{\partial t^2 \partial \phi} \Phi_1 \Big] / [-2 \hbar^2 (\sin(\theta))^2 (f(r))^2 r^2], \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\phi\theta} = \mathcal{G}_{\theta\phi} = & \left[-\hbar^2 f(r) r \frac{\partial^3}{\partial \phi^3} \Phi_2 + \hbar^2 r^3 (\sin(\theta))^2 \frac{\partial^3}{\partial t^2 \partial \phi} \Phi_2 + \hbar^2 r^3 (\sin(\theta))^4 \frac{\partial^3}{\partial \theta \partial t^2} \Phi_3 \right. \\
& - 4 \hbar^2 f(r) (\sin(\theta))^2 \frac{\partial^2}{\partial \theta \partial \phi} \Phi_1 - 4 \hbar^2 r^2 (\sin(\theta))^2 (f(r))^2 \frac{\partial^2}{\partial r \partial \phi} \Phi_2 \\
& - 4 \hbar^2 r^2 (\sin(\theta))^4 \left(\frac{\partial^2}{\partial \theta \partial r} \Phi_3 \right) (f(r))^2 \\
& + 2 \hbar^2 r^3 (\sin(\theta))^3 \left(\frac{\partial^2}{\partial t^2} \Phi_3 \right) \cos(\theta) + 5 \hbar^2 r (\sin(\theta))^4 \left(\frac{\partial}{\partial \theta} \Phi_3 \right) f(r) \\
& - 2 \hbar^2 r (\sin(\theta))^4 \left(\frac{\partial}{\partial \theta} \Phi_3 \right) (f(r))^2 - 2 \hbar^2 (f(r))^2 r (\sin(\theta))^2 \\
& \times \frac{\partial}{\partial \phi} \Phi_2 + 4 \hbar^2 f(r) \sin(\theta) \cos(\theta) \frac{\partial}{\partial \phi} \Phi_1 + 3 \hbar^2 f(r) r (\cos(\theta))^2 \frac{\partial}{\partial \phi} \Phi_2 \\
& - \hbar^2 f(r) r (\sin(\theta))^4 \frac{\partial^3}{\partial \theta^3} \Phi_3 - \hbar^2 r^3 (\sin(\theta))^2 (f(r))^2 \frac{\partial^3}{\partial r^2 \partial \phi} \Phi_2 \\
& - \hbar^2 r^3 (\sin(\theta))^4 (f(r))^2 \frac{\partial^3}{\partial \theta \partial r^2} \Phi_3 + \left(\frac{\partial^3}{\partial \theta \partial r \partial \phi} \Phi_1 \right) f(r) r (\sin(\theta))^2 \hbar^2 \\
& + \left(\frac{\partial^3}{\partial \theta \partial t \partial \phi} \Phi_0 \right) f(r) r (\sin(\theta))^2 \hbar^2 - \hbar^2 f(r) r (\sin(\theta))^2 \frac{\partial}{\partial \phi} \Phi_2 \\
& - m^2 f(r) r^3 (\sin(\theta))^4 \frac{\partial}{\partial \theta} \Phi_3 - m^2 f(r) r^3 (\sin(\theta))^2 \frac{\partial}{\partial \phi} \Phi_2 \\
& + 4 \hbar^2 r (\sin(\theta))^3 \Phi_3 f(r) \cos(\theta) \\
& \left. - \hbar^2 r^3 (\sin(\theta))^4 f(r) \times \left(\frac{d}{dr} f(r) \right) \frac{\partial^2}{\partial \theta \partial r} \Phi_3 - \hbar^2 r^3 (\sin(\theta))^2 \right]
\end{aligned}$$

$$\begin{aligned}
 & \times f(r) \left(\frac{d}{dr} f(r) \right) \frac{\partial^2}{\partial r \partial \phi} \Phi_2 + \hbar^2 f(r) r (\sin(\theta))^2 \left(\frac{\partial}{\partial \theta} \Phi_3 \right) (\cos(\theta))^2 \\
 & - 2 \hbar^2 r^3 (\sin(\theta))^3 (f(r))^2 \times \left(\frac{\partial^2}{\partial r^2} \Phi_3 \right) \cos(\theta) \\
 & - 5 \hbar^2 f(r) r (\sin(\theta))^3 \left(\frac{\partial^2}{\partial \theta^2} \Phi_3 \right) \cos(\theta) - 3 \hbar^2 f(r) \cos(\theta) r \left(\frac{\partial^2}{\partial \theta \partial \phi} \Phi_2 \right) \sin(\theta) \\
 & + 2 \hbar^2 f(r) r \sin(\theta) \left(\frac{\partial^2}{\partial \phi^2} \Phi_3 \right) \cos(\theta) + 4 \hbar^2 r (\sin(\theta))^3 \Phi_3 (f(r))^2 \cos(\theta) \\
 & - 4 \hbar^2 r^2 (\sin(\theta))^3 \left(\frac{\partial}{\partial r} \Phi_3 \right) (f(r))^2 \cos(\theta) + 8 \hbar^2 f(r) \sin(\theta) r (\cos(\theta))^3 \Phi_3 \\
 & - 2 m^2 f(r) r^3 (\sin(\theta))^3 \Phi_3 \cos(\theta) - 2 \hbar^2 r^3 (\sin(\theta))^3 f(r) \\
 & \times \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial r} \Phi_3 \right) \cos(\theta) \Big] / [-\hbar^2 f(r) (\sin(\theta))^2 r], \tag{A.9}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{G}_{tr} = \mathcal{G}_{rt} = & \left[-\hbar^2 r^3 (\sin(\theta))^2 \frac{\partial^3}{\partial t^3} \Phi_1^2 \hbar (f(r))^2 (\sin(\theta))^2 r^2 \frac{\partial^2}{\partial \theta \partial t} \Phi_2 \right. \\
 & + 2 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{\partial}{\partial t} \Phi_1 \right) \left(\frac{d}{dr} f(r) \right)^2 + 2 \hbar^2 (f(r))^2 (\sin(\theta))^2 \\
 & \times \left(\frac{\partial}{\partial t} \Phi_1 \right) r - 2 \hbar^2 (f(r))^4 (\sin(\theta))^2 \left(\frac{\partial}{\partial r} \Phi_0 \right) r \\
 & - 2 \hbar^2 \sin(\theta) \cos(\theta) (f(r))^3 \frac{\partial}{\partial \theta} \Phi_0 - 2 \hbar^2 (f(r))^2 (\sin(\theta))^2 r^2 \frac{\partial^2}{\partial t \partial r} \Phi_1 \\
 & + \hbar^2 (f(r))^3 (\sin(\theta))^2 \left(\frac{\partial^3}{\partial \theta^2 \partial r} \Phi_0 \right) r - \hbar^2 f(r) (\sin(\theta))^2 r \frac{\partial^3}{\partial \theta^2 \partial t} \Phi_1 \\
 & + \hbar^2 r^3 (\sin(\theta))^2 (f(r))^4 \frac{\partial^3}{\partial r^3} \Phi_0 + \left(\frac{\partial^3}{\partial t \partial r \partial \phi} \Phi_3 \right) \\
 & \times (f(r))^2 r^3 (\sin(\theta))^2 \hbar^2 + \left(\frac{\partial^3}{\partial \theta \partial t \partial r} \Phi_2 \right) (f(r))^2 r^3 (\sin(\theta))^2 \hbar^2 \\
 & + \hbar^2 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial^2}{\partial \phi^2} \Phi_1 \right) (f(r))^2 r - m^2 f(r) r^3 (\sin(\theta))^2 \frac{\partial}{\partial t} \Phi_1 \\
 & + m^2 (f(r))^3 r^3 (\sin(\theta))^2 \frac{\partial}{\partial r} \Phi_0 + 2 \hbar^2 r^2 (\sin(\theta))^2 \left(\frac{\partial^2}{\partial t \partial \phi} \Phi_3 \right) (f(r))^2 \\
 & + 2 \hbar^2 (f(r))^4 (\sin(\theta))^2 r^2 \frac{\partial^2}{\partial r^2} \Phi_0 + \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d^3}{dr^3} f(r) \right) \Phi_0 (f(r))^3 \\
 & + \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) f(r) \frac{\partial^2}{\partial t^2} \Phi_0 - \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \\
 & \times \left(\frac{\partial^2}{\partial t \partial r} \Phi_1 \right) f(r) + \hbar^2 r \sin(\theta) \cos(\theta) (f(r))^3 \left(\frac{\partial^2}{\partial \theta \partial r} \Phi_0 \right) \\
 & - \hbar^2 f(r) \sin(\theta) \cos(\theta) r \frac{\partial^2}{\partial \theta \partial t} \Phi_1 - \hbar^2 r^3 f(r) (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^3 \Phi_0 \\
 & + \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right)^2 (f(r))^2 \frac{\partial}{\partial r} \Phi_0 + m^2 (f(r))^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \\
 & \times \Phi_0 + \hbar^2 (f(r))^2 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial^2}{\partial \theta^2} \Phi_0 \right) r \\
 & \left. + \hbar^2 r^3 (\sin(\theta))^2 f(r) \left(\frac{\partial}{\partial t} \Phi_1 \right) \frac{d^2}{dr^2} f(r) + 4 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) (f(r))^3 \right]
 \end{aligned}$$

$$\begin{aligned}
& \times \frac{\partial^2}{\partial r^2} \Phi_0 + 2 \hbar^2 (f(r))^3 (\sin(\theta))^2 r^2 \left(\frac{d^2}{dr^2} f(r) \right) \Phi_0 \\
& + 4 \hbar^2 (f(r))^3 (\sin(\theta))^2 r^2 \left(\frac{d}{dr} f(r) \right) \frac{\partial}{\partial r} \Phi_0 + 2 \hbar^2 f(r) (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) r^2 \\
& \times \frac{\partial}{\partial t} \Phi_1 - 2 \hbar^2 (f(r))^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \Phi_0 r \\
& + 2 \hbar^2 \sin(\theta) \cos(\theta) r^2 \left(\frac{\partial}{\partial t} \Phi_2 \right) (f(r))^2 + 3 \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d^2}{dr^2} f(r) \right) \\
& \times (f(r))^3 \frac{\partial}{\partial r} \Phi_0 - 2 \hbar^2 (f(r))^3 \frac{\partial^2}{\partial \phi^2} \Phi_0 \\
& + \hbar^2 r^3 (\sin(\theta))^2 \left(\frac{d}{dr} f(r) \right) \left(\frac{d^2}{dr^2} f(r) \right) \Phi_0 (f(r))^2 \\
& + \hbar^2 \sin(\theta) \cos(\theta) \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial \theta} \Phi_0 \right) (f(r))^2 r \\
& + \hbar^2 (f(r))^3 \left(\frac{\partial^3}{\partial r \partial \phi^2} \Phi_0 \right) r - \hbar^2 f(r) r \frac{\partial^3}{\partial t \partial \phi^2} \Phi_1 \\
& - 2 \hbar^2 (f(r))^3 (\sin(\theta))^2 \frac{\partial^2}{\partial \theta^2} \Phi_0 \Big] / [(f(r))^2 r^3 (\sin(\theta))^2 \hbar^2]. \tag{A.10}
\end{aligned}$$

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