Effect of Lorentz symmetry breaking on the deflection of light in a cosmic string spacetime

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We investigate the Lorentz symmetry breaking effects (LSBE) on the deflection of light by a rotating cosmic string spacetime in the weak limit approximation. We first calculate the deflection angle by a static cosmic string for a fixed spacelike 4-vector case (FSL) with the corresponding effective-string optical metric using the Gauss-Bonnet theorem (GBT). Then, we focus on a more general scenario, namely we calculate the deflection angle by a rotating cosmic string applying the GBT to Randers effective-string metric. We obtain a significant modification in the deflection angle because of the LSBE parameter. We find first and second order correction terms due to the global effective topology which are proportional to the cosmic string and LSBE parameter, respectively. Finally, for a fixed time-like 4-vector (FTL) case, we show that the deflection angle is not affected by LSBE parameter.

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I. INTRODUCTION

Today, gravitational lensing phenomenon, which studies the effects of light deflection on the appearance of cosmic objects is a useful theoretical tool in observational cosmology [1]. Since Eddington's famous observed eclipse [2], the topic of gravitational lensing has gained more attention. One remarkable fact about the gravitational field is that the deflection angle depends neither on the nature of the matter nor on its physical state. Moreover, light deflection is an indirect perception to compute the total matter density around a black hole [3]. Besides, gravitational lensing can also shed light on the possible existence and properties of topological defects that are formed during the phase transitions in the early universe. The examples of those topological defects are monopoles, cosmic strings and domain walls [4,5].

Recently, behavior of a relativistic spin-0 particles that are subject to a scalar potential under the effects of the Lorentz symmetry breaking [6–10] in the cosmic string spacetime has been studied by Bakke *et al.* [11,12]. To tackle the problem, they have considered two possible scenarios of the anisotropy generated by a Lorentz symmetry breaking effect (LSBE). They defined the LSBE term by using a tensor $(\mathcal{K}_F)_{\mu\nu\alpha\beta}$. This tensor governs the Lorentz symmetry violation in the *CPT* [C: charge conjugation, P: parity, and T: time reversal]-even gauge sector of the standard model extension [13]. Using the scalar potential, which modifies the mass term in the Klein- Gordon equation (KGE), it has been shown that the cosmic string spacetime can be changed with the effects of the Lorentz symmetry violation backgrounds.

An effective geometrical method for computing the gravitational lensing of a considered black hole was introduced by Gibbons and Werner (GW) [14,15]. GW's method calculates the asymptotic deflection angle by employing the Gauss-Bonnet theorem (GBT). Moreover, the method of GW was extended to the stationary metrics by Werner. Thus, he managed to get the deflection angle for the Kerr black hole whose optical geometry is Finslerian [16]. Today, there exists numerous papers in the literature that use the GW's method. Along this line of thinking, for example, the deflection angles of static cosmic strings and global monopoles were studied by Jusufi [17,18]. Furthermore, recently the deflection angle for the infrared region by using the Gaussian curvature of the optical metric of Rindler modified Schwarzschild back hole has been investigated by Sakalli and Ovgun [19] in which the role of the Rindler acceleration on the gravitational lensing is neatly shown. Meanwhile, it is worth noting that by applying a complex coordinate transformation, Newman and Janis [20] established a relationship between the nonrotating and rotating spacetimes of general relativity.

In this paper, we use the cosmic string spacetime with the LSBE [11,12] to analyze the light deflection predicted by Einstein's general theory of relativity. To this end, we follow the method of GW [14,15]. Thus, by integrating the Gaussian curvature of the optical metric outwards from the light ray, we plan to reveal how the LSBE plays a

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role on the cosmic string spacetime and modifies the deflection angle.

The paper is organized as follows. In Sec. II, we briefly review the FSL 4-vector case of the cosmic string metric of [11,12]. The second part of this section is devoted to the derivation of effective Gaussian curvature within the GBT and its corresponding deflection angle. In Sec. III, we study the deflection angle of the rotating cosmic string metric with LSBE. In particular we consider the effective Gaussian optical curvature and deflection angle. In Sec. IV, we briefly review and calculate the deflection angle for a fixed FTL 4-vector case of a static cosmic string with the LSBE. We extend our results and investigate the deflection angle for a rotating cosmic string for a FTL 4-vector case. In Sec. V, we consider the geodesics equations to recover the deflection angles. Finally, we draw our conclusions in Sec. VI. Throughout this paper, we shall use natural units, i.e. $G = c = \hbar = 1$.

II. EFFECTIVE COSMIC STRING METRIC FOR A FSL 4-VECTOR CASE

A. A static cosmic string for a FSL 4-vector case

The Lagrange density for the non-birefringent modified Maxwell theory coupled to gravity [11,21,22] is given by

$$\mathcal{L}_{\rm mod} = -\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + \frac{1}{4}\mathcal{K}^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}\right), \quad (1)$$

where $\mathcal{K}^{\mu\nu\rho\lambda}$ is a Lorentz symmetry violating tensor, which guarantees the CPT symmetry. $\mathcal{K}^{\mu\nu\rho\lambda}$ shares the same features of the Riemann tensor, plus some additional double-traceless conditions:

$$\mathcal{K}_{\mu\nu\rho\lambda} = \mathcal{K}_{[\mu\nu][\rho\lambda]}, \qquad \mathcal{K}_{\mu\nu\rho\lambda} = \mathcal{K}_{\rho\lambda\mu\nu}, \qquad \mathcal{K}^{\mu\nu}{}_{\mu\nu} = 0. \quad (2)$$

By using the following effective metric tensor

$$g^{\rm eff}_{\mu\rho}(x) = g_{\mu\rho}(x) + \epsilon\xi_\mu\xi_\rho, \quad (3)$$

in which the parameter ϵ is governed by $\epsilon = \kappa/(1+\kappa\xi_\rho\xi^\rho/2)$ with $0 \leq \kappa < 2$, one can see (as being stated in [11]) that $g^{\mu\nu}(x)$ background attributes an anisotropy, which means that the propagation of light must be modified by the background. Let us first consider a normalized parameter four-vector ξ_a as a spacelike 4-vector:

$$\xi_a = (0, 0, 1, 0). \quad (4)$$

Under the Lorentz symmetry breaking, a topological defect in curved spacetime can be expressed by the effective metric tensor of the cosmic string [23], whose line-element in cylindrical coordinates is given by the effective metric [11]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \eta^2\rho^2(1+\epsilon)\mathrm{d}\varphi^2 + \mathrm{d}z^2. \quad (5)$$

In the above equation η is the parameter of the cosmic string. Moreover, η is expressed by $\eta = 1-4\mu$, where μ is the linear mass density of the cosmic string. We can easily write the above metric in spherical coordinates. To do so, let us introduce the following coordinates transformations $z = r\cos\theta$ and $\rho = r\sin\theta$. Thus, metric (5) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \eta^2 r^2(1+\epsilon)\sin^2\theta\mathrm{d}\varphi^2, \quad (6)$$

then we can find the corresponding optical metric form of metric (6), if we first project the metric into equilateral plane with $\theta = \pi/2$ and immediately after we consider the null case: $\mathrm{d}s^2 = 0$. Therefore, one gets the optical metric of the line-element (6) as follows

$$\mathrm{d}t^2 = \mathrm{d}r^2 + \eta^2 r^2(1+\epsilon)\mathrm{d}\varphi^2. \quad (7)$$

We now introduce a new coordinate $r^\star$, thereby a new function $f(r^\star)$:

$$\mathrm{d}r^\star = \mathrm{d}r, \qquad f(r^\star) = \eta r\sqrt{1+\epsilon}. \quad (8)$$

Therefore, the optical metric (7) becomes [14]

$$\mathrm{d}t^2 = \tilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}r^{\star 2} + f^2(r^\star)\mathrm{d}\varphi^2. \quad (9)$$

To derive the corresponding effective Gaussian optical curvature K of the metric (9), we follow [14]:

$$K = -\frac{1}{f(r^\star)}\frac{\mathrm{d}^2 f(r^\star)}{\mathrm{d}r^{\star 2}}, \\ = -\frac{1}{f(r^\star)}\left[\frac{\mathrm{d}r}{\mathrm{d}r^\star}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}r^\star}\right)\frac{\mathrm{d}f}{\mathrm{d}r} + \left(\frac{\mathrm{d}r}{\mathrm{d}r^\star}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right]. \quad (10)$$

Since $f(r^\star)$ is linear in r, one can check by using Eq. (10) that in fact the effective Gaussian optical vanishes, i.e. $K = 0$. This result will provide us some convenience during the computation of the light deflection in the following sections.

B. Effective Gaussian curvature and deflection angle

The GBT for the nonsingular region D_R in M, with boundary $\partial D_R = \gamma_{\tilde{g}} \cup C_R$ can be stated as follows [14]

$$\iint_{\mathcal{D}_R} K\mathrm{d}S + \oint_{\partial\mathcal{D}_R} \kappa\mathrm{d}t + \sum_i \theta_i = 2\pi\chi(\mathcal{D}_R). \quad (11)$$

Note that κ is the geodesic curvature, K gives the Gaussian optical curvature, θ_i gives the corresponding exterior angle at the i^{th} vertex, and $\chi(\mathcal{D}_R)$ is the Euler characteristic number. The geodesic curvature can be computed as $\kappa = \tilde{g}(\nabla_{\dot{\gamma}}\dot{\gamma}, \ddot{\gamma})$, in which the unit speed condition holds $\tilde{g}(\dot{\gamma}, \dot{\gamma}) = 1$ and $\ddot{\gamma}$ is the unit acceleration vector.

While $R \to \infty$, both jump angles (θ_O, θ_S) become $\pi/2$, and hence $\theta_O + \theta_S \to \pi$; in which the subscripts S and O correspond to the source and observer, respectively (see for example [14]). Furthermore, since $\gamma_{\tilde{g}}$ is a geodesic, then it follows $\kappa(\gamma_{\tilde{g}}) = 0$. Let us find now the geodesic curvature, which can be calculated as $\kappa(C_R) = |\nabla_{\dot{C}_R}\dot{C}_R|$ in which one can choose $C_R \coloneqq r(\varphi) = R = \text{const}$. The radial component of the geodesic curvature can be calculated as

$$(\nabla_{\dot{C}_R}\dot{C}_R)^r = \dot{C}^{\varphi}_R(\partial_\varphi \dot{C}^r_R) + \tilde{\Gamma}^r_{\varphi\varphi}(\dot{C}^{\varphi}_R)^2. \tag{12}$$

The first term vanishes, while the second term gives nonzero contribution. To see this, we should recall the nonzero component $\tilde{\Gamma}^r_{\varphi\varphi} = -f(r^\star)f'(r^\star)$ and the unit speed condition $\tilde{g}_{\varphi\varphi}\dot{C}^{\varphi}_R\dot{C}^{\varphi}_R = 1$, where $f(r^\star)$ is given by Eq. (8). Using these relations and the optical metric (9), it follows immediately that

$$\mathrm{d}t = \eta R\sqrt{1+\epsilon}\,\mathrm{d}\varphi. \tag{13}$$

Thus, for very large R, the geodesic curvature reads

$$\begin{aligned}\lim_{R\to\infty}\kappa(C_R) &= \lim_{R\to\infty}|\nabla_{\dot{C}_R}\dot{C}_R|,\\ &= \lim_{R\to\infty}\left(\frac{\eta^2(1+\epsilon)}{\eta^2R^2(1+\epsilon)}\right)^{1/2},\\ &\to \frac{1}{R},\end{aligned} \tag{14}$$

which suggests that

$$\kappa(C_R)\mathrm{d}t = \eta\sqrt{1+\epsilon}\,\mathrm{d}\varphi. \tag{15}$$

Reconsidering Eq. (11) and recalling that the Euler characteristic is characterized by $\chi(\mathcal{D}_R) = 1$, we find

$$\iint_{\mathcal{D}_R} K\mathrm{d}S + \oint_{C_R}\kappa\mathrm{d}t \overset{R\to\infty}{=} \iint_{\mathcal{D}_\infty} K\mathrm{d}S + \eta\sqrt{1+\epsilon}\int_0^{\pi+\hat{\alpha}}\mathrm{d}\varphi = \pi, \tag{16}$$

in which the domain $\mathcal{D}_\infty$ connotes an infinite domain bounded by the light ray $\gamma_{\tilde{g}}$. Thus, the asymptotic deflection angle $\hat{\alpha}$ can be found as

$$\hat{\alpha} = \frac{\pi}{\eta\sqrt{1+\epsilon}} - \pi. \tag{17}$$

Using Taylor series in η and ϵ, we can approximate the result for the deflection angle as

$$\hat{\alpha} \simeq 4\mu\pi - \frac{\epsilon\pi}{2} - 2\pi\mu\epsilon + \mathcal{O}(\mu^2, \eta^2). \tag{18}$$

The first term is just the deflection angle by a static cosmic string. Interestingly, due to the Lorentz symmetry breaking by the parameter ϵ, we find that the deflection angle decreases.

III. ROTATING COSMIC STRINGS WITH FSL 4-VECTOR

A. Effective String-Randers optical metric

Let us introduce a rotating cosmic string by using the following coordinate transformations [24,25]

$$\mathrm{d}t \to \mathrm{d}t + a\mathrm{d}\varphi, \tag{19}$$

into the metric (5), we find

$$\mathrm{d}s^2 = -(\mathrm{d}t + a\mathrm{d}\varphi)^2 + \mathrm{d}\rho^2 + \eta^2\rho^2(1+\epsilon)\mathrm{d}\varphi^2 + \mathrm{d}z^2. \tag{20}$$

Let us now introduce the tetrads $e^a{}_\mu(x)$, which satisfies the relation $g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\mu(x)\eta_{ab}$, in which η_{ab} is the Minkowski tensor. In particular we choose the tetrads as follows

$$e^a{}_\mu(x) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ a & 0 & \eta\rho\sqrt{1+\epsilon} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{21}$$

Next, by writing the four vector $\xi_\mu(x)$, in terms of tetrads as $\xi_\mu(x) = e^a{}_\mu(x)\xi_a$, and choosing $\xi_a = (\zeta, \sigma, \gamma, \delta)$, one can show that $\xi_\mu(x)\xi^\mu(x) = -\zeta^2 + \sigma^2 + \gamma^2 + \delta^2 = \text{const}$ holds for the rotation case. In particular, the choice of $\xi_a = (0, 0, 1, 0)$ results in $\xi_\mu(x)\xi^\mu(x) = 1 = \text{const}$. Metric (20) can be expressed in spherical coordinates as follows

$$\mathrm{d}s^2 = -(\mathrm{d}t + a\mathrm{d}\varphi)^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \eta^2r^2\alpha\sin^2\theta\mathrm{d}\varphi^2, \tag{22}$$

where we have introduced $\alpha = 1 + \epsilon$. The stationary metric can be recasted to give a Finslerian optical metric of Randers type with the Hessian given as [16]

$$g_{ij}(x, X) = \frac{1}{2}\frac{\partial^2F^2(x, X)}{\partial X^i\partial X^j}. \tag{23}$$

Furthermore, by homogeneity we have $F^2(x, X) = g_{ij}(x, X)X^iX^j$, thus the Randers metric can also be written in the following from [15]

$$F(x,X) = \sqrt{a_{ij}(x)X^iX^j} + b_i(x)X^i, \tag{24}$$

where a_{ij} and b_i must satisfy the condition $a^{ij}b_ib_j < 1$.

One can easily find the corresponding Randers optical metric for our stationary effective cosmic string spacetime by writing the stationary spacetime (22) as [15]

$$\mathrm{d}s^2 = V^2[-(\mathrm{d}t - b_i\mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j], \tag{25}$$

where

$$a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \eta^2 r^2\alpha\sin^2\theta\mathrm{d}\varphi^2, \tag{26}$$

$$b_i(x)\mathrm{d}x^i = -a\mathrm{d}\varphi^2. \tag{27}$$

In this paper we shall consider a planar light ray by setting $\theta = \pi/2$, then making use of Eqs. (24) and (25), we end up with the effective Randers-string metric:

$$F\left(r,\varphi,\frac{\mathrm{d}r}{\mathrm{d}t},\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) = \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \eta^2r^2\alpha\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2} - a\frac{\mathrm{d}\varphi}{\mathrm{d}t}. \tag{28}$$

Hence the Randers metric for null geodesics $\mathrm{d}s^2 = 0$, gives $\mathrm{d}t = F(x,\mathrm{d}x)$. On the other hand Fermat's principle suggests that light rays γ are selected by the following condition

$$0 = \delta\int_\gamma \mathrm{d}t = \delta\int_{\gamma_F} F(x,\dot{x})\mathrm{d}t, \tag{29}$$

where it is important to note that these spatial light rays γ are also geodesics γ_F of the Randers metric F. This is crucial point since we can apply the so-called Nazım's method [26] to construct a Riemannian manifold $(\mathcal{M},\bar{g})$, osculating the Randers manifold $(\mathcal{M},F)$. To do so, we choose a smooth and nonzero vector field $\bar{X}$ over $\mathcal{M}$ (except at single vertex points) with $\bar{X}(\gamma_F) = \dot{x}$. The Hessian (23) then reads

$$\bar{g}_{ij}(x) = g_{ij}(x,\bar{X}(x)). \tag{30}$$

It is quite remarkable fact that the geodesic γ_F of $(\mathcal{M},F)$ is also a geodesic $\gamma_{\bar{g}}$ of $(\mathcal{M},\bar{g})$ i.e., $\gamma_F = \gamma_{\bar{g}}$ (see [16] for details). Hence we shall use the cosmic string effective optical metric and construct the corresponding osculating Riemannian manifold $(\mathcal{M},\bar{g})$. This is important since it allows us to calculate the deflection angle of the planar light ray. We choose the line $r(\varphi) = b/\sin\varphi$, where b is known as the impact parameter and gives the minimal radial distance of the light ray from the cosmic string lying along the z axis. We make the following choose for the leading terms of the vector field $\bar{X} = (\bar{X}^r,\bar{X}^\varphi)(r,\varphi)$ (see for details [16])

$$\bar{X}^r = -\cos\varphi + \mathcal{O}(a), \qquad \bar{X}^\varphi = \frac{\sin^2\varphi}{b} + \mathcal{O}(a). \tag{31}$$

In the next section we shall use these relationsto calculate the Gaussian curvature and apply the GBT theorem to the osculating Riemannian optical metric.

B. Deflection angle

In this section we shall apply the GBT to the osculating optical geometries. In particular one can apply the GBT to a domain $(\mathcal{D}_R,\bar{g})$ within the region $\mathcal{D}_R$ possessing boundary curve $\partial\mathcal{D}_R = \gamma_{\bar{g}} \cup C_R$ [16]

$$\iint_{\mathcal{D}_R} K\mathrm{d}S + \oint_{\partial\mathcal{D}_R}\kappa\mathrm{d}t + \sum_i\theta_i = 2\pi\chi(\mathcal{D}_R). \tag{32}$$

Note that in a similar way we define the geodesic curvature as $\kappa = |\nabla_{\dot{\gamma}}\dot{\gamma}|$ (with respect to $\bar{g}$). Hereinbefore, as $R\to\infty$ the both jump angles tends to $\pi/2$, hence in an analogous way as in the last section we have $\theta_O + \theta_S \to \pi$. Thus, Eq. (32) recasts in

$$\iint_{\mathcal{D}_R} K\mathrm{d}S + \oint_{\partial\mathcal{D}_R}\kappa\mathrm{d}t = 2\pi\chi(\mathcal{D}_R) - (\theta_O + \theta_S) = \pi. \tag{33}$$

Since the geodesic curvature in the case of geodesics $\gamma_{\bar{g}}$ vanishes i.e. $\kappa(\gamma_{\bar{g}}) = 0$, we shall now focus on calculating $\kappa(C_R)\mathrm{d}t$ where $\kappa(C_R) = |\nabla_{\dot{C}_R}\dot{C}_R|$. For very large but constant R given by $C_R \coloneqq r(\varphi) = R = \text{const}$, if we use the unit speed condition i.e. $\bar{g}_{\varphi\varphi}\dot{C}_R^\varphi\dot{C}_R^\varphi = 1$, and the nonzero Christoffel symbol $\bar{\Gamma}^r_{\varphi\varphi}$, the geodesic curvature is found to be $\kappa(C_R) \to R^{-1}$. The effective cosmic string-optical metric (28) then gives

$$\mathrm{d}t = \left(\sqrt{\eta^2R^2\alpha} - a\right)\mathrm{d}\varphi. \tag{34}$$

Hence if we combine these results it follows

$$\lim_{R\to\infty}\kappa(C_R)\mathrm{d}t = \lim_{R\to\infty}\left(\sqrt{\eta^2\alpha} - \frac{a}{R}\right) = \eta\sqrt{\alpha}\mathrm{d}\varphi. \tag{35}$$

We clearly see that our effective optical metric is not asymptotically Euclidean i.e. $\kappa(C_R)\mathrm{d}t/\mathrm{d}\varphi = \eta\sqrt{\alpha} \neq 1$, due to the fact that our spacetime metric (22) is globally conical. Clearly if we set $\eta\sqrt{\alpha} \to 1$, we find the asymptotically Euclidean case, i.e. $\kappa(C_R)\mathrm{d}t/\mathrm{d}\varphi = 1$. To find the deflection angle we first approximate the boundary curve of $\mathcal{D}_\infty$ by a notional undeflected ray, that is, the line $r(\varphi) = b/\sin\varphi$, then GBT reduces to

$$\hat{\alpha} \simeq \pi\left(\frac{1}{\eta\sqrt{\alpha}} - 1\right) - \frac{1}{\eta\sqrt{\alpha}}\int_0^\pi\int_{\frac{b}{\sin\varphi}}^\infty K\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi. \tag{36}$$

We can make use of the Eqs. (23), (33), and (35), to find the following relations

$$\bar{g}_{rr} = 1 - \frac{\sin^6\varphi\alpha\eta^2 r^2 a}{b^3(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{3/2}} + \mathcal{O}(a^2), \tag{37}$$

$$\bar{g}_{\varphi\varphi} = r^2\eta^2\alpha - \frac{a\sin^2\varphi\alpha r^2(2r^2\eta^2\alpha\sin^4\varphi + 3b^2\cos^2\varphi)\eta^2}{b^3(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{3/2}} \tag{38}$$

$$\bar{g}_{r\varphi} = \frac{a\cos^3\varphi}{b^3(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{3/2}} + \mathcal{O}(a^2), \tag{39}$$

neglecting higher order terms of the angular momentum parameter a. Then the determinant of this metric can be written as

$$\det\bar{g} = r^2\eta^2\alpha - \frac{3a\sin^2\varphi\alpha\eta^2 r^2(\sin^4\varphi\alpha\eta^2 r^2 + \cos^2\varphi b^2)}{b^3(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{3/2}}. \tag{40}$$

For the Christoffel symbols we find,

$$\bar{\Gamma}^{\varphi}_{rr} = -\frac{3a\sin^4\varphi[2(-\sin\varphi r + b)\cos^2\varphi - r\sin^3\varphi]\cos\varphi}{2rb^3(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{5/2}},$$

$$\bar{\Gamma}^{\varphi}_{r\varphi} = \frac{1}{r} + \frac{a\alpha\eta^2 r\sin^6\varphi[2\sin^4\varphi\alpha\eta^2 r^2 + 5b^2\cos^2\varphi]}{2(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{5/2}b^5}. \tag{41}$$

The Gaussian curvature is

$$K = \frac{\bar{R}_{r\varphi r\varphi}}{\det\bar{g}} = \frac{1}{\sqrt{\det\bar{g}}}\left[\frac{\partial}{\partial\varphi}\left(\frac{\sqrt{\det\bar{g}}}{\bar{g}_{rr}}\bar{\Gamma}^{\varphi}_{rr}\right) - \frac{\partial}{\partial r}\left(\frac{\sqrt{\det\bar{g}}}{\bar{g}_{rr}}\bar{\Gamma}^{\varphi}_{r\varphi}\right)\right], \tag{42}$$

so using the Christoffel symbols and the metric components, we obtain

$$K = -\frac{12a}{r}f(r,\varphi,\eta,\alpha), \tag{43}$$

with

$$f(r,\varphi,\eta,\alpha) = \frac{\sin^3\varphi}{(\cos^2\varphi + \frac{r^2\eta^2\alpha\sin^4\varphi}{b^2})^{7/2}b^7}\left[-\frac{\sin^{11}\varphi\alpha^3\eta^6 r^5}{24} + \frac{b^2 r^3\eta^2\alpha\sin^9\varphi}{8} + \frac{r^3\eta^2\alpha\cos^2\varphi b^2(\eta^2\alpha + 27)\sin^7\varphi}{24}\right.$$
$$- \frac{3b^3 r^2\eta^2\alpha\cos^2\varphi\sin^6\varphi}{4} + \left(\frac{5b^2 r^3\eta^2\alpha\cos^4\varphi}{4} - \frac{\cos^2\varphi b^4 r}{2}\right)\sin^5\varphi - \frac{3r^2 b^3\alpha\eta^2\cos^4\varphi\sin^4\varphi}{2} \tag{44}$$
$$\left. + \frac{17r(\eta^2\alpha - \frac{33}{17})\cos^4\varphi b^4\sin^3\varphi}{24} + \frac{\cos^4\varphi\sin^2\varphi b^5}{2} - \frac{5b^4 r\sin\varphi\cos^6\varphi}{4} + b^5\cos^6\varphi\right]. \tag{45}$$

The deflection angle (36) reduces to

$$\hat{\alpha} \simeq 4\pi\mu - \frac{\epsilon\pi}{2} - 2\mu\epsilon\pi - \frac{1}{\eta\sqrt{\alpha}} \times \int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\left(-\frac{12a}{r}f(r,\varphi,\eta,\alpha)\right)\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi, \tag{46}$$

in which b is the impact parameter. After we integrate with respect to the radial coordinate and considering a Taylor expansion around μ and ϵ, we find a nonzero contribution for a retrograde light ray

$$\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\frac{12a}{r}f(r,\varphi,\eta,\alpha)\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi = \frac{3a\pi\mu}{2b} - \frac{3a\pi\epsilon}{16b} - \frac{3a\pi\epsilon\mu}{4b} + \mathcal{O}(\mu^2,\epsilon^2). \tag{47}$$

But zero contribution for the prograde light ray

$$\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\frac{12a}{r}f(r,\varphi,\eta,\alpha)\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi = 0. \tag{48}$$

Finally the deflection angle for the retrograde case gives

$$\hat{\alpha}_{\mathrm{ret}} \simeq 4\pi\mu - \frac{\epsilon\pi}{2} - 2\pi\mu\epsilon + \frac{3a\pi\mu}{2b} - \frac{3a\pi\epsilon}{16b}, \tag{49}$$

and similarly for the prograde case

$$\hat{\alpha}_{\mathrm{prog}} \simeq 4\pi\mu - \frac{\epsilon\pi}{2} - 2\pi\mu\epsilon. \tag{50}$$

It is interesting to note that since the rotating cosmic string parameter a is proportional to the angular momentum, i.e. $a = 4J$, which contains the mass per unit length, μ, by definition of the angular momentum. As a consequence, the last two terms in Eq. (49) can be considered as second order terms, more precisely, the term $a\mu$ can be viewed as μ^2, while the last term can also be viewed as μ^2, is we assume μ and ϵ, say, to be of the same order of magnitude. In this way, if we neglect these terms, we end up with the following result

$$\hat{\alpha} \simeq 4\pi\mu - \frac{\epsilon\pi}{2} - 2\pi\mu\epsilon. \tag{51}$$

IV. EFFECTIVE COSMIC STRING METRIC FOR A FTL 4-VECTOR CASE WITH LSBE

A. Static cosmic strings

In this section we shall consider a normalized parameter four-vector ξ_a as a timelike 4-vector given by

$$\xi_a = (1, 0, 0, 0). \tag{52}$$

In this case, under the LSB, a cosmic string metric can be expressed by the effective metric [23], whose line-element in cylindrical coordinates is given by [6]

$$\mathrm{d}s^2 = -(1-\epsilon)\mathrm{d}t^2 + \mathrm{d}\rho^2 + \eta^2\rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \tag{53}$$

Introducing a spherically coordinates transformations to the above metric and considering the equatorial plane, for the optical metric it follows

$$\mathrm{d}t^2 = \frac{\mathrm{d}r^2}{(1-\epsilon)} + \frac{\eta^2 r^2 \mathrm{d}\varphi^2}{(1-\epsilon)}. \tag{54}$$

We now introduce a new coordinate $r^\star$, thereby a new function $f(r^\star)$:

$$\mathrm{d}r^\star = \frac{\mathrm{d}r}{\sqrt{1-\epsilon}}, f(r^\star) = \frac{\eta r}{\sqrt{1-\epsilon}}. \tag{55}$$

Moreover, we show that the corresponding Gaussian optical curvature vanishes also in this case i.e. $K = 0$. We need to apply the GBT but first let us see that for very large R, the optical metric gives

$$\mathrm{d}t = \frac{\eta R}{\sqrt{1-\epsilon}}\mathrm{d}\varphi. \tag{56}$$

Using the optical metric (49), one can show in a similar way the following relation for the geodesic curvature $\kappa(C_R) \to R^{-1}\sqrt{1-\epsilon}$, and consequently $\kappa(C_R)\mathrm{d}t = \eta\mathrm{d}\varphi$. From the GBT we find

$$\iint_{D_R} K\mathrm{d}S + \oint_{C_R} \kappa \mathrm{d}t \overset{R\to\infty}{=} \iint_{S_\infty} K\mathrm{d}S + \eta\int_0^{\pi+\hat{\alpha}} \mathrm{d}\varphi = \pi. \tag{57}$$

Hence, for the deflection angle the last equation we derive the following result

$$\hat{\alpha} \simeq 4\mu\pi + \mathcal{O}(\mu^2). \tag{58}$$

As expected, the LSBE parameter for a FTL 4-vector case is not relevant for light deflection, which is different from the FSL 4-vector case.

B. Rotating cosmic strings

In a similar way as in the FSL 4-vector case, we shall introduce a rotating cosmic string into the effective metric (53). In that case we find

$$\mathrm{d}s^2 = -(1-\epsilon)(\mathrm{d}t + a\mathrm{d}\varphi)^2 + \mathrm{d}\rho^2 + \eta^2\rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \tag{59}$$

Now we will try to show that the condition $\xi_\mu(x)\xi^\mu(x) = \mathrm{const}$ is indeed satisfied. To do so, let us first choose the following tetrads for our metric (59)

$$e^a{}_\mu(x) = \begin{pmatrix} \sqrt{1-\epsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a\sqrt{1-\epsilon} & 0 & \eta\rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{60}$$

Then by writing the four vector $\xi_\mu(x) = e^a{}_\mu(x)\xi_a$, and choosing for generally $\xi_a = (\zeta, \sigma, \gamma, \delta)$, one can show that $\xi_\mu(x)\xi^\mu(x) = -\zeta^2 + \sigma^2 + \gamma^2 + \delta^2 = \mathrm{const}$. Furthermore if we choose, $\xi_a = (1, 0, 0, 0)$, we find $\xi_\mu(x)\xi^\mu(x) = -1 = \mathrm{const}$.

Thus, by considering the spherical coordinate transformations, we find

$$\mathrm{d}s^2 = -\beta(\mathrm{d}t + a\mathrm{d}\varphi)^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \eta^2 r^2 \sin^2\theta \mathrm{d}\varphi^2 \tag{61}$$

where $\beta = 1 - \epsilon$. This leads to the considerably simpler Randers type metric

$$F\left(r, \varphi, \frac{\mathrm{d}r}{\mathrm{d}t}, \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) = \sqrt{\frac{1}{\beta}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\eta^2 r^2}{\beta}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2} - a\frac{\mathrm{d}\varphi}{\mathrm{d}t}. \tag{62}$$

The effective Gaussian curvature gives:

$$K = -\frac{12a}{r} f(r, \varphi, \eta, \beta) \tag{63}$$

where

$$f(r,\varphi,\eta,\beta)=\frac{\sin^3\varphi}{(\frac{\cos^2\varphi}{\beta}+\frac{r^2\eta^2\sin^4\varphi}{b^2\beta})^{7/2}b^7\beta^2}\left[-\frac{\sin^{11}\varphi\eta^6r^5}{24}+\frac{b^2r^3\eta^2\sin^9\varphi}{8}+\frac{r^3\eta^2\cos^2\varphi b^2(\eta^2+27)\sin^7\varphi}{24}\right.$$
$$-\frac{3b^3r^2\eta^2\cos^2\varphi\sin^6\varphi}{4}+\left(\frac{5b^2r^3\eta^2\cos^4\varphi}{4}-\frac{\cos^2\varphi b^4r}{2}\right)\sin^5\varphi-\frac{3r^2b^3\eta^2\cos^4\varphi\sin^4\varphi}{2} \tag{64}$$
$$\left.+\frac{17r(\eta^2-\frac{33}{17})\cos^4\varphi b^4\sin^3\varphi}{24}+\frac{\cos^4\varphi\sin^2\varphi b^5}{2}-\frac{5b^4r\sin\varphi\cos^6\varphi}{4}+b^5\cos^6\varphi\right]. \tag{65}$$

We can now integrate with respect to the radial coordinate first and then make a Taylor series expansion around μ and ϵ, for the retrograde light ray we find

$$\int_0^\pi\int_{\frac{b}{\sin\varphi}}^\infty\frac{12a}{r}f(r,\varphi,\eta,\beta)\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi=\frac{3\pi a\mu}{2b}\left(1-\frac{\epsilon}{2}\right). \tag{66}$$

While for the prograde light ray we find zero contribution

$$\int_0^\pi\int_{\frac{b}{\sin\varphi}}^\infty\frac{12a}{r}f(r,\varphi,\eta,\beta)\sqrt{\det\bar{g}}\mathrm{d}r\mathrm{d}\varphi=0. \tag{67}$$

For the total deflection angle, we find the following results for the retrograde case

$$\hat{\alpha}_{\mathrm{ret}}=4\mu\pi+\frac{3\pi a\mu}{2b}(1-\epsilon), \tag{68}$$

and for the prograde case

$$\hat{\alpha}_{\mathrm{prog}}=4\mu\pi. \tag{69}$$

Hence, since the rotating cosmic string parameter a is proportional to the angular momentum J, the last two terms in Eq. (68) can be considered as second order terms μ^2. Therefore, by neglecting these terms we find

$$\hat{\alpha}=4\mu\pi. \tag{70}$$

V. GEODESICS EQUATIONS

A. Effective metric for FSL 4-vector case

We can apply now the variational principle $\delta\int\mathcal{L}\mathrm{d}s=0$, to calculate the deflection angle in the stationary spacetime metric (22). The Lagrangian can be written as [27]

$$\mathcal{L}=-\frac{1}{2}(\dot{t}+a\dot{\varphi})^2+\frac{\dot{r}^2}{2}+\frac{1}{2}r(s)^2(\dot{\theta}^2+\eta^2\alpha\sin^2\theta\dot{\varphi}^2) \tag{71}$$

We can simplify further this problem by choosing $\theta=\pi/2$. Next, we introduce two constants of motion, say l and γ, given by

$$p_\varphi=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}=-(\dot{t}+a\dot{\varphi})a+\eta^2\alpha r(s)^2\dot{\varphi}=l \tag{72}$$

$$p_t=\frac{\partial\mathcal{L}}{\partial\dot{t}}=-(a\dot{\varphi}+\dot{t})=-\gamma. \tag{73}$$

We change the coordinates by using $r=1/u(\varphi)$, which leads to the following identity

$$\frac{\dot{r}}{\dot{\varphi}}=\frac{\mathrm{d}r}{\mathrm{d}\varphi}=-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}. \tag{74}$$

Moreover, we make clear that the angle φ is measured from the point of closest approach i.e. $u=u_{\max}=1/r_{\min}=1/b$ [28]. Hence, we can choose for the first and second constant $\gamma=1$ and $l=\eta\sqrt{\alpha}b$, respectively. This leads to the following equation

$$\frac{1}{2u^4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2+\frac{\eta^2\alpha}{2u^2}-\frac{1}{2}\frac{(\eta^2\alpha-a^2u^2-\eta\sqrt{\alpha}bu^2a)^2}{u^4(a+\eta\sqrt{\alpha}b)^2}-\frac{a(\eta^2\alpha-a^2u^2-\eta\sqrt{\alpha}bu^2a)}{u^2(a+\eta\sqrt{\alpha}b)}-\frac{a^2}{2}=0. \tag{75}$$

We can solve the above differential equation (75) using a perturbation method. The solution of this differential equation in leading order terms can be written in the form

$$\Delta\varphi=\pi+\hat{\alpha}, \tag{76}$$

where $\hat{\alpha}$ is the deflection angle. Solving for $\mathrm{d}u/\mathrm{d}\varphi$ [30], one can show that the above result can be written as

$$\hat{\alpha}=2|\varphi(u_{max})-\varphi_\infty|-\pi. \tag{77}$$

where we have used a Taylor series expansion around μ, a, and ϵ, which leads to the following integral

$$\varphi=\int_0^{1/b}A(u,\mu,a,\varepsilon,b)\,\mathrm{d}u \tag{78}$$

The function $A(u,\mu,a,\epsilon,b)$ is given by

$$A(u,\mu,a,\epsilon,b) = \frac{b[2a(\epsilon-1)(8\mu+1)+\zeta]}{2\sqrt{(1-b^2u^2)b^2(b^2u^2-1)}}, \tag{79}$$

where

$$\zeta = ((4\mu+1)\epsilon - 8\mu - 2) - (\epsilon-2)(4\mu+1)u^2b^3.$$

We find the following result for the deflection angle in leading order terms

$$\hat{\alpha} \simeq 4\mu\pi - \frac{\epsilon\pi}{2} - 2\pi\epsilon\mu. \tag{80}$$

Thus, we have recovered the deflection angle which corresponds to the static case given by Eq. (51).

B. Effective metric for FTL 4-vector case

Using a similar approach we can calculate the deflection angle in the stationary spacetime metric (54). The Lagrangian can be written as

$$\mathcal{L} = -\frac{\beta}{2}(\dot{t}+a\dot{\varphi})^2 + \frac{\dot{r}^2}{2} + \frac{1}{2}r(s)^2(\dot{\theta}^2+\eta^2\sin^2\theta\dot{\varphi}^2). \tag{81}$$

Then in the equatorial plane we find

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = -\beta(\dot{t}+a\dot{\varphi})a + \eta^2 r(s)^2\dot{\varphi} = l \tag{82}$$

$$p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -\beta(a\dot{\varphi}+\dot{t}) = -\gamma. \tag{83}$$

Without loss of generality, we can choose $\gamma = 1$ and $l = (\eta b)/\sqrt{\beta}$. The following differential equation can be obtained

$$\frac{1}{2u^4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + \frac{\eta^2}{2u^2} - \frac{1}{2}\frac{(\eta^2-a^2\beta u^2-\eta\sqrt{\beta}bu^2a)^2}{u^4\beta(a+(\eta b)/\sqrt{\beta})^2} - \frac{a(\eta^2-a^2\beta u^2-\eta\sqrt{\beta}bu^2a)}{(a+(\eta b)/\sqrt{\beta})} - \frac{a^2\beta}{2} = 0. \tag{84}$$

This result leads to the following integral

$$\varphi = \int_0^{1/b} B(u,\mu,a,\epsilon,b)\,\mathrm{d}u, \tag{85}$$

where the function $B(u,\mu,a,\epsilon,b)$ is given by

$$B(u,\mu,a,\epsilon,b) = \frac{b[a(\epsilon-2)(8\mu+1)+\Xi]}{2\sqrt{(1-b^2u^2)b^2(b^2u^2-1)}}, \tag{86}$$

where

$$\Xi = b(2u^2b^2(4\mu+1) - 2b(4\mu+1)).$$

Working in a similar fashion as in the FSL 4-vector case, the solution of Eq. (84) is written as

$$\Delta\varphi = \pi + \hat{\alpha}, \tag{87}$$

with the deflection angle $\hat{\alpha}$ given by

$$\hat{\alpha} \simeq 4\mu\pi. \tag{88}$$

This result is consistent with the Eq. (70) found by the GB method.

VI. CONCLUSION

In this paper, we have first computed the deflection angle by virtue of a cosmic string having the LSBE. To this end, we have applied the GBT to the effective-optical metric of the static cosmic string spacetime for a FSL 4-vector case. The first term of the deflection angle (18) was found to be the ordinary deflection angle by a static cosmic string. However, it has been shown that Lorentz symmetry breaking, which is parametrized by ϵ decreases the deflection angle. On the other hand, for a FTL 4-vector case with the LSBE, the deflection angle remains unchanged, remarkably. Next, we have extended our results to rotating cosmic strings. We have derived their corresponding both Randers type effective-cosmic string optical metrics and the effective Gaussian optical curvatures. We then have constructed the osculating Riemannian manifolds and applied the GBT to those obtained optical metrics. We have deduced from our results that the deflection angle is not affected by the rotation of the cosmic string in leading order terms. Thus, the results obtained by the GB method are in perfect agreement with the geodesics computations in the leading order terms.

It is worth noting that the main outcome of this study is that LSBE plays an important role on the light deflection in a cosmic string spacetime. From this point of view, the latter remark is the newest contribution to the subject of gravitational lensing. Besides, LSBE can be considered in the future observations about gravitational lensing. Moreover, GB method is a powerful theoretical technique for finding an exact result of the deflection angle. Because, it evaluates the associated deflection angle integral in the domain that connotes an infinite domain bounded by the light ray. In conclusion, GB method reveals that light deflection can be seen as a *partially topological effect* of the spacetime geometry, which is very recently discussed in [29].

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[1] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
[2] M. Longair, Phil. Trans. R. Soc. A **373**, 20140287 (2015).
[3] M. Bartelmann and P. Schneider, Phys. Rep. **340**, 291 (2001).
[4] N. Sakai and D. Tong, J. High Energy Phys. 03, 019 (2005).
[5] K. Jusufi, M. C. Werner, A. Banerjee, and A. Ovgun, Phys. Rev. D **95**, 104012 (2017).
[6] H. Belich, C. Furtado, and K. Bakke, Eur. Phys. J. C **75**, 410 (2015).
[7] K. Bakke and H. Belich, J. Phys. G **42**, 095001 (2015).
[8] K. Bakke and H. Belich, Ann. Phys. (Berlin) **354**, 1 (2015).
[9] H. Belich and K. Bakke, Phys. Rev. D **90**, 025026 (2014).
[10] A. G. de Lima, H. Belich, and K. Bakke, Rev. Math. Phys. **28**, 1650023 (2016).
[11] H. F. Mota and K. Bakke, Phys. Rev. D **89**, 027702 (2014).
[12] H. Belich and K. Bakke, Int. J. Mod. Phys. A **31**, 1650026 (2016).
[13] R. Casana, E. S. Carvalho, and M. M. Ferreira Jr, Phys. Rev. D **84**, 045008 (2011).
[14] G. W. Gibbons and M. C. Werner, Classical Quantum Gravity **25**, 235009 (2008).
[15] G. W. Gibbons, C. A. R. Herdeiro, C. Warnick, and M. C. Werner, Phys. Rev. D **79**, 044022 (2009).
[16] M. C. Werner, Gen. Relativ. Gravit. **44**, 3047 (2012).
[17] K. Jusufi, Astrophys. Space Sci. **361**, 24 (2016).
[18] K. Jusufi, Eur. Phys. J. C **76**, 332 (2016).
[19] I. Sakalli and A. Ovgun, arXiv:1702.04636.
[20] E. T. Newman and A. I. Janis, J. Math. Phys. (N.Y.) **6**, 915 (1965).
[21] G. Betschart, E. Kant, and F. R. Klinkhamer, Nucl. Phys. **B815**, 198 (2009).
[22] E. Kant, F. R. Klinkhamer, and M. Schreck, Phys. Lett. B **682**, 316 (2009).
[23] A. Vilenkin, E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 2000).
[24] P. O. Mazur, Phys. Rev. Lett. **57**, 929 (1986).
[25] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984); J. R. Gott and M. Alpert, Gen. Relativ. Gravit. **16**, 243 (1984); S. Giddings, J. Abbot, and K. Kuchar, Gen. Relativ. Gravit. **16**, 751 (1984); M. S. Cunha, C. R. Muniz, H. R. Christiansen, and V. B. Bezerra, Eur. Phys. J. C **76**, 512 (2016).
[26] A. N. Nazım Terzioğlu, Über Finslersche Räume (Wolf, München, 1936); A. N. Terzioğlu, Über den Satz von Gauss-Bonnet im Finslerschen Raum, in *Université d'Istanbul, Faculté des Sciences:Recueil de Mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences.* (Kenan Matbaası, Istanbul, 1948), p. 26.
[27] R. H. Boyer and R. W. Lindquist, J. Math. Phys. (N.Y.) **8**, 265 (1967).
[28] L. Iorio, Nuovo Cimento Soc. Ital. Fis. **118B**, 249 (2003).
[29] I. Fernandez-Nunez and O. Bulashenko, Phys. Lett. A **381**, 1764 (2017).
[30] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, New York, 1972).