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Abstract: We study light rays in the static and spherically symmetric gravitational field of the null aether theory (NAT). To this end, we employ the Gauss-Bonnet theorem to compute the deflection angle formed by a NAT black hole in the weak limit approximation. Using the optical metrics of the NAT black hole, we first obtain the Gaussian curvature and then calculate the leading terms of the deflection angle. Our calculations indicate how gravitational lensing is affected by the NAT field. We also illustrate that the bending of light stems from global and topological effects.

Keywords: deflection of light, Gauss-Bonnet theorem, gravitational lensing, black hole, null aether theory

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1 Introduction

According to the theory of general relativity (GR) [1], gravity affects space-time geometry, and the gravitational field produced by matter can be powerful enough to drastically alter the ordinary causal structure of space-time and produce a region, the so-called black hole (BH), in which even light is held and no part can escape to infinity. Therefore, scientists have been studying BHs in detail over the years [2-6].

The event horizon of a BH is the limit to which events cannot impact an observer on the opposite side. That is, it is the gravitational point of no return, and nobody can really "see" a BH. In fact, the event horizon is a null surface that separates the rays of light that reach infinity from those falling into singularity. On the other hand, astronomers can observe this curvature in space-time in the form of light from distant objects that gets bent while going around objects in the foreground. In particular, if a BH is immersed in a bright region, like a disc of glowing gas, it creates a dark region similar to a shadow [7, 8]. This shadow, caused by gravitational bending and capture of light by the event horizon, reveals a lot about the nature of these massive objects. From this information, the position and mass of the BH can be determined. For static and spherical symmetric spaces, suitable coordinate systems can be introduced to determine the location of the event horizon by considering the points at which the local light cones tilt over. This means that the existence of the event horizon (i.e., BH) is related to the local Lorentz invariance of time. Therefore, it is of significance to study the properties of BHs in the context of gravity theories that address the local Lorentz invariance violations. For example, in GR, the existence of BHs depends on the causal structure, which originates from both the Lorentz symmetry in matter fields and the local flatness theorem. Therefore, it is natural to examine whether or not BHs will still be formed in the absence of Lorentz symmetry; if they exist, it is even more essential to analyze their physical features, as the signatures of Lorentz violations appear in the regime of strong gravity. On the other hand, when the Lorentz symmetry is broken and the causal structure is modified in the most radical way, BH solutions surprisingly do exist. In these new BH solutions, the event horizon is replaced by the universal horizon to be able to capture any mode independently of the propagation velocity [9, 10]. Some BH solutions have already been found in restricted Lorentz-violating gauges, such as spherical symmetry [9, 11] and slowly rotating backgrounds, both in lower dimensions [12] and four dimensions [13-16]. Generally, numerical solutions to the equations of Hořava gravity and the Einstein-aether theory admit such BHs. Nevertheless, analytical solutions are rare for such theories. However, these solutions are obtained in symmetry-restricted scenarios for various asymptotics [17]. In the Einstein-aether theory, the vec-

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tor field is timelike everywhere, and it explicitly breaks the boost sector of Lorentz symmetry, which has been studied extensively in the literature (see, for instance, [18]). In particular, aether models present themselves as phenomenological probes to test for the presence of Lorentz symmetry breaking (LSB) in astrophysical objects and cosmology. Aether models are nothing but vector inputs to the Lagrangian density of a system having a nonvanishing vacuum expectation. Because of this property, the vector field dynamically selects an opted frame at each point in space-time and automatically breaks the Lorentz invariance. This is a mechanism that resembles the breaking of local symmetry in the Higgs mechanism [19] and serves as a phenomenological representation of the LSB terms in the gravitation sector of the Standard Model Extension [20].

The null aether theory (NAT) is one of the new vector-tensor theories of the modified gravity theory [21]. In this theory, the dynamical vector field acts as the aether, and an exact spherically symmetric BH solution with charge is possible [22]. It was also discussed in [22] that the NAT fields affect the solar system dynamics when the Eddington-Robertson-Schiff parameters β and γ, which appear in the perihelion precession and light deflection expressions, are extracted for the NAT BHs. Furthermore, it was shown that in the post-Newtonian order, although there is no contribution from the NAT field to the deflection of light rays passing near the BH (as in GR), the NAT field contributes to the perihelion precession of planetary orbits. Later remark can play a role on solar system observations. The physical features (singularity structure, ADM mass, and thermodynamics) of the NAT BH are also analyzed in [22]. Moreover, the NAT charge is able to reduce the horizon thermodynamics to that of the Reissner-Nordström-(A)dS BH of GR and modify the circular orbits of massive and massless particles around the BH. To utilize the differential deflection exhibited by weak lensing, the first weak deflection angle is calculated. It depends on the mass distribution of the gravitational lensing system. Gibbons and Werner reported that it is possible to calculate the deflection angle in weak field limits using the Gauss-Bonnet theorem (GBT) and optical geometry [23, 24]. In this method, they focus on the domain outside the trajectory of light. The optical metric has geodesics, which are spatial light rays, such that the focusing of light rays is considered as a topological effect [25]. At the present time, the GW method has been applied to various space-time metrics of black holes and wormholes (see, for example, [26-61] and the references therein). We can define the domain surface as (D,χ,g) using the Euler characteristic χ and a Riemannian metric g. Thus, the GBT can be formulated as follows [23]:

$$\int\int_D K\mathrm{d}S + \int_{\partial D}\kappa\mathrm{d}t + \sum_i \alpha_i = 2\pi\chi(D), \tag{1}$$

where α_i is the exterior angle with the ith vertex, K represents the Gaussian curvature, and κ represents the geodesics curvature. This method only works for asymptotically flat space-times and is expressed as follows [23]:

$$\hat{\alpha} = -\int\int_D K\mathrm{d}S. \tag{2}$$

In this study, our main purpose is to explore the effects of NAT on gravitational lensing. To this end, we organize the paper as follows: In Section 2, we briefly review the BH space-time of the NAT. Section 3 is devoted to the computation of the deflection angle by NAT BH using the GBT in a weak field regime and in the plasma medium. We conclude our results in Section 4. Natural units are used throughout this paper: $\hbar = c = 1$.

2 NAT BH space-time

The line element of the asymptotically flat NAT BH is expressed as [22]

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2, \tag{3}$$

where

$$h(r) = \begin{cases} 1 - \dfrac{2a_1^2 b_1}{r^{1+q}} - \dfrac{2a_2^2 b_2}{r^{1-q}} - \dfrac{2\tilde{m}}{r} & (\text{for } q \neq 0), \\ 1 - \dfrac{2m}{r} & (\text{for } q = 0), \end{cases} \tag{4}$$

where a_1, a_2, $\tilde{m}$, and m are integration constants and

$$q \equiv \sqrt{9 + 8\frac{c_1}{c_{23}}},$$
$$b_1 = \frac{1}{8}[c_3 - 3c_2 + c_{23}q], \quad b_2 = \frac{1}{8}[c_3 - 3c_2 - c_{23}q], \tag{5}$$

in which c_1, c_2, c_3, and $c_{23} = c_2 + c_3$ are dimensionless constant parameters. Furthermore, the constants $\tilde{m}$ and m are the mass parameters of the solutions.

As is evident from Eq. (4), in the case $q = 0$, the metric is nothing but the well-known Schwarzschild space-time, which is asymptotically flat. However, in the case $q \neq 0$, to achieve asymptotically flat boundary conditions, one should consider the following cases separately (by definition, $q > 0$ [22]):

$$h(r)|_{r\to\infty} = 1 \begin{cases} \text{for } 0 < q < 1 \\ (\text{if } a_1 \neq 0 \text{ and } a_2 \neq 0) \text{ or } (\text{if } a_1 = 0 \text{ or } b_1 = 0), \\ \\ \text{for } 0 < q \\ (\text{if } a_2 = 0 \text{ or } b_2 = 0). \end{cases} \tag{6}$$

In this study, we shall consider the case of $a_2 = 0$.

Thus, the metric function $h(r)$ and the scalar aether field $\phi(r)$ take the following forms:

$$h(r) = 1 - \frac{2a_1^2 b_1}{r^{1+q}} - \frac{2\tilde{m}}{r}, \tag{7}$$

$$\phi(r) = \frac{a_1}{r^{(1+q)/2}}. \tag{8}$$

The location of the event horizon r_0 is given by $h(r_0) = 0$ and the area of the event horizon is $A = 4\pi r_0^2$. Setting $a_1 = GQr_0^{(q-1)/2}$, where Q is the NAT "charge," Eqs. (7) and (8) become

$$h(r) = 1 - \frac{2G^2Q^2 b_1}{r^2}\left(\frac{r_0}{r}\right)^{q-1} - \frac{2\tilde{m}}{r}, \tag{9}$$

$$\phi(r) = \frac{GQ}{r}\left(\frac{r_0}{r}\right)^{(q-1)/2}. \tag{10}$$

At the location r_0, we have

$$h(r_0) = 1 - \frac{2G^2Q^2 b_1}{r_0^2} - \frac{2\tilde{m}}{r_0} = 0, \tag{11}$$

$$\phi(r_0) = \frac{GQ}{r_0}. \tag{12}$$

It is worth noting that the horizon condition (11) is independent of the parameter q. In addition, the scalar aether field $\phi(r)$ resembles the electric potential at $r = r_0$.

Using the asymptotically flat solutions of the NAT black hole given in [22], taking $q = 1$, the metric function and scalar aether field are expressed as

$$h(r) = 1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}, \tag{13}$$

$$\phi(r) = \frac{a_1}{r^{1/2}}. \tag{14}$$

The deflection angle of photon can be calculated using the following formula (r_0 is the distance of closest approach) [62]:

$$\hat{\alpha}(r_0) = -\pi + 2\int_{r_0}^{\infty} \mathrm{d}r \frac{1}{r\sqrt{\frac{r^2 f(r_0)}{r_0^2} - h(r)}}. \tag{15}$$

However, in most cases, it is not easy to solve this integral. For example, in the case of $\frac{\tilde{m}}{r_0} \ll 1$, the deflection angle is found to be too small, which is known as weak lensing. Moreover, $\hat{\alpha}$ grows as r_0 approaches the photosphere until it diverges and converts into strong lensing.

3 Calculation of weak deflection angle of NAT BH

3.1 Weak deflection angle and GBT

In this section, we calculate the weak deflection angle of NAT BH using the GBT. First, for simplicity, we assume that $\theta = \pi/2$ for the equatorial plane and use the space-time metric given in Eq. (13) to write the optical metric as follows:

$$\mathrm{d}t^2 = \frac{\mathrm{d}r^2}{\left(1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}\right)^2} + \frac{r^2}{\left(1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}\right)}\mathrm{d}\phi^2. \tag{16}$$

Next, we calculate the Gaussian curvature of the optical NAT BH space-time:

$$K = \frac{R}{2} \approx -2\frac{\tilde{m}}{r^3} + 6\frac{b_1(2\tilde{m} - r)a_1{}^2}{r^5}. \tag{17}$$

Here, using the above Gaussian curvature of the optical NAT BH space-time in the GBT, we obtain the deflection angle. The GBT provides the relation between the intrinsic geometry of space-time and the topology of the region D_R in M, with the boundary $\partial D_R = \gamma_{\tilde{g}} \cup C_R$ [23]:

$$\int_{D_R} K\,\mathrm{d}S + \oint_{\partial D_R} \kappa\,\mathrm{d}t + \sum_i \epsilon_i = 2\pi\chi(D_R). \tag{18}$$

Note that $\kappa = \tilde{g}(\nabla_{\dot{\gamma}}\dot{\gamma}, \ddot{\gamma})$represents the geodesic curvature, where $\tilde{g}(\dot{\gamma}, \dot{\gamma}) = 1$ and $\ddot{\gamma}$ is the unit acceleration vector, and ϵ_i corresponds to the exterior angle at the i^{th} vertex. As $r \to \infty$, both jump angles reduce to $\pi/2$, and it is found that $\theta_O + \theta_S \to \pi$. Because D_R is not singular, the Euler characteristic is $\chi(D_R) = 1$. Hence, the GBT is expressed as

$$\iint_{D_R} K\,\mathrm{d}S + \oint_{\partial D_R} \kappa\,\mathrm{d}t + \theta_i = 2\pi\chi(D_R), \tag{19}$$

in which $\gamma_{\tilde{g}}$ is a geodesic and $\theta_i = \pi$ denotes the total jump angle. Therefore, we have $\kappa(\gamma_{\tilde{g}}) = 0$. After recalling the Euler characteristic number, which is $\chi = 1$, we find that the remaining part yields $\kappa(C_R) = |\nabla_{\dot{C}_R}\dot{C}_R|$ as $r \to \infty$. The radial component of the geodesic curvature is calculated as follows:

$$\left(\nabla_{\dot{C}_R}\dot{C}_R\right)^r = \dot{C}_R^{\varphi}\partial_{\varphi}\dot{C}_R^r + \Gamma_{\varphi\varphi}^r\left(\dot{C}_R^{\varphi}\right)^2. \tag{20}$$

At very large R, $C_R := r(\varphi) = r = \text{const}$, and we have

$$\left(\nabla_{\dot{C}_R^r}\dot{C}_R^r\right)^r \to -\frac{1}{r}. \tag{21}$$

It is noted that the geodesic curvature does not depend on topological defects, $\kappa(C_R) \to r^{-1}$. Thereafter, from the optical space-time metric (16), one can see that $\mathrm{d}t = r\,\mathrm{d}\varphi$; thus,

$$\kappa(C_R)\mathrm{d}t = \mathrm{d}\varphi. \tag{22}$$

Using the above results, the GBT equation reduces to the following form:

$$\iint_{D_R} K\,\mathrm{d}S + \oint_{C_R} \kappa\,\mathrm{d}t \overset{r\to\infty}{=} \iint_{S_\infty} K\,\mathrm{d}S + \int_0^{\pi+\hat{\alpha}} \mathrm{d}\varphi. \tag{23}$$

In the weak deflection limit, one may assume that the light ray is expressed as $r(t) = u/\sin\varphi$ at the zeroth order. Thereafter, we use the straight line approximation [23]

$r = u/\sin\phi$, where u is the impact parameter, and Eq. (2) becomes

$$\hat{\alpha} = -\int_0^{\pi}\int_{\frac{u}{\sin\phi}}^{\infty} K\mathrm{d}S, \tag{24}$$

where $\mathrm{d}S = r\mathrm{d}r\mathrm{d}\phi$. Note that we ignore the higher order terms. Hence, Eqs. (17) and (24) are simplified to the following expression for the deflection angle of the NAT BH in the second order owing to weak lensing:

$$\hat{\alpha} \simeq 3/2\,\frac{{a_1}^2 b_1 \pi}{u^2} + 4\,\frac{\tilde{m}}{u}, \tag{25}$$

where the deflection angle is in full agreement with Eq. (115) in the paper [22]. Here, one realizes that depending on the sign of the aether field parameter b_1, the light deflection can be more or less than the GR value expressed by the first term in Eq. (25). For $b_1 < 0$, the aether field decreases the light deflection angle relative to the Schwarzschild case in GR. This is similar to the effect of charge in the Reissner-Nordström solution [63, 64] for the weak-field limits. Thus, in the presence of the $b_1 > 0$ NAT parameter, the aether increases the deflection angle, and the deflection angle reduces to that in the case of the Schwarzschild BH when $b_1 = 0$. The deflection angle in the leading order terms is seen to be in agreement with [22].

3.2 Weak deflection angle of NAT BH in plasma medium

To consider the effects of plasma [38], in this subsection we shall examine the case in which light travels from vacuum to a hot, ionized gas medium. Let v be the velocity of light through the plasma. Then, the refractive index $n(r)$ is expressed as follows:

$$n(r) \equiv \frac{c}{v} = \frac{1}{\mathrm{d}r/\mathrm{d}t} \qquad \{\because c = 1\}. \tag{26}$$

Thus, we obtain the refractive index $n(r)$ for a NAT BH: [38]

$$n(r) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2}\left(1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}\right)}, \tag{27}$$

where ω_e and ω_∞ are the electron plasma and photon frequencies measured by an observer at infinity, respectively. The line element (13) can be rewritten as

$$\mathrm{d}\sigma^2 = g_{ij}^{\mathrm{opt}}\mathrm{d}x^i\mathrm{d}x^j = \frac{n^2(r)}{1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}}\left[\frac{\mathrm{d}r^2}{1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}} + r^2\mathrm{d}\phi^2\right]. \tag{28}$$

The optical Gaussian curvature can then be expressed as

$$K \approx -2\,\frac{\tilde{m}}{r^3} - 3\,\frac{\tilde{m}{\omega_e}^2}{{\omega_\infty}^2 r^3} + \left(-6\,r^{-4} + 12\,\frac{\tilde{m}}{r^5} + \left(-10\,\frac{1}{{\omega_\infty}^2 r^4} + 52\,\frac{\tilde{m}}{{\omega_\infty}^2 r^5}\right){\omega_e}^2\right){a_1}^2 b_1. \tag{29}$$

On the other hand, it follows from Eq. (28) that

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right|_{C_r} = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2}\left(1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}\right)}\left(\frac{r^2}{1 - \frac{2a_1^2 b_1}{r^2} - \frac{2\tilde{m}}{r}}\right)^{1/2}, \tag{30}$$

and we have

$$\lim_{R\to\infty} \kappa_g \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right|_{C_R} = 1. \tag{31}$$

For the limit $R \to \infty$ and using the straight light approximation $r = u/\sin\varphi$, the GBT is then expressed as [38]

$$\lim_{R\to\infty}\int_0^{\pi+\alpha}\left[\kappa_g \frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right]\Bigg|_{C_R}\mathrm{d}\varphi = \pi - \lim_{R\to\infty}\int_0^{\pi}\int_{\frac{u}{\sin\varphi}}^{R}\mathcal{K}\mathrm{d}S. \tag{32}$$

Consequently, the deflection angle yields

$$\hat{\alpha} \approx 6\,\frac{\tilde{m}{\omega_e}^2}{u{\omega_\infty}^2} + 4\,\frac{\tilde{m}}{u} + \frac{5{a_1}^2 b_1 {\omega_e}^2 \pi}{2u^2{\omega_\infty}^2} + \frac{3a^2 b_1 \pi}{2u^2}, \tag{33}$$

where the photon rays are moving in a medium of homogeneous plasma. Note that in the absence of plasma ($\omega_e = 0$) or ($\omega_e/\omega_\infty \to 0$), this deflection angle reduces to the vacuum case calculated in Eq. (25). It is clear that for the photons propagating in a homogeneous plasma for the case of frequency $\omega_e/\omega_\infty = 6\times10^{-3}$ [65], the deflection angle increases. However, the effect of the plasma medium cannot be detected easily owing to its small value in near-future observations.

4 Conclusions

In this study, we have examined the weak gravitational lensing of NAT BH, which is a solution of the new vector-tensor theory. After integrating the deflection angle integral (24), analytically, we have illustrated that if $\frac{\tilde{m}}{r_0} \ll 1$, the deflection angle becomes too small. The latter remark is an evidence of weak lensing. Remarkably, $\hat{\alpha}$ increases as r_0 approaches the photosphere until it diverges to produce strong lensing. The aether field parameter b_1 modifies the gravitational lensing in such a way that when $b_1 < 0$, the aether field decreases the light deflection angle relative to the Schwarzschild BH of GR. This result is analogous to the effect of charge in the Reissner-Nordström BH [63, 64] in the weak-field limit. On the other hand, the positive NAT parameter increases the deflection angle, which reduces to the case of the Schwarzschild BH when $b_1 = 0$. Further, in the existence of plasma ($\omega_e = 0$), the photons propagate in a homogeneous plasma for the case of frequency $\omega_e/\omega_\infty = 6\times10^{-3}$

[65-71], and the deflection angle increases. However, it seems that the effect of the plasma medium will not be detected in the near future owing to its infinitesimal value [72].

In future work, we plan to study our topological framework on the gravitational lensing of rotating NAT BHs, which can be obtained using the Newman-Janis algorithm [73], similar to the Kerr and/or BTZ algorithms, which are accepted as more realistic BH geometries. Spraying the particles from their ergosphere affects the moving photons around the rotating BH's photon sphere. Therefore, it will be interesting to analyze the deflection angle of the rotating NAT BH. We believe that the results to be obtained will shed light on future observations.

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