Effect of the dilaton field and plasma medium on deflection angle by black holes in Einstein-Maxwell-dilaton-axion theory

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In this paper, we argue that one can calculate the weak deflection angle of light in the background of an Einstein-Maxwell-Dilaton-axion black hole using the Gauss-Bonnet theorem. To support this, the optical geometry of the black hole and the Gibbons-Werner method is used to obtain the deflection angle of light in the weak-field limits. Moreover, we investigate the effect of a plasma medium on the deflection of light for a given black hole. Because the dilaton and axion are candidates for dark matter, this can give us a hint about the observation of the dark matter that is supported by the black hole. Hence, we demonstrate the observational viability by showing the effect of dark matter on the weak deflection angle of light.

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I. INTRODUCTION

Since the first photo of the Messier 87 black hole by the Event Horizon Telescope, studying black holes has received quite a lot of attention [1]. On the other hand, the Laser Interferometer Gravitational-wave Observatory (LIGO) observed yet another enormous collision in space, and this one seems to be between a black hole and a neutron star [2]. As fascinating as it is mysterious, dark matter is one of the greatest enigmas of astrophysics and cosmology. Recently, the existence of dark matter was shown by Di Paolo *et al.*, disproving the empirical relations describing alternative theories [3]. Moreover, the ongoing estimations by the WMAP mission [4] continue to describe the relative abundance of dark and baryonic matter in our Universe with incredible accuracy. Moreover, most of the matter-energy content of the Universe consists of cold dark matter, whose composition is still obscure. The most promising theoretical particle proposed to solve the enigma of dark matter is the *axion* [5,6]. This particle was first described in 1977 by Peccei and Quinn (PQ) [7] in their attempt to describe the strong-*CP* problem in QCD theory. In spite of the fact that the first PQ axion is currently prohibited, other axion models are still reasonable. The hypothesis that the axion might be the dark matter particle has been widely discussed [8–12].

String theory is a candidate for a consistent theory of quantum gravity, and obviously the attributes of black holes (BHs) in string theory are most intriguing. String theory differs from general relativity due to the existence of a scalar field known as the dilaton field, which causes changes in the characteristics of BH geometries. In spite of the dilaton, the axion has been accepted as a strong candidate for dark matter by numerous physicists, and experiments to distinguish it have been effectively performed [13–15], while the dilaton has not been identified thus far, despite its theoretical significance. It is important to mention that the dilaton produces a fifth force which can influence Einstein's gravity in a fundamental way [16,17]. Also, in cosmology it can assume the part of the inflation, furthermore, can be an amazing competitor of the dark matter [13–18]. There are solid tests and hypothetical proof for the presence of dark matter in the Universe from gravitational lensing, galactic revolution curves, and well-known inflationary models. The study will also offer new insights into the nature of the dilaton field and axion dark matter.

It is a well-known fact that gravitational lensing is an effective method to search for not only dark and massive objects, but also wormholes and BHs. Gravitational lensing is a specific effect of light deflection. The gravitational bending of light by mass prompted the first exploratory confirmations of the general theory of relativity. The bending of light has theoretical significance, particularly for examining the null structure of a spacetime. Gravitational lensing occurs due to the warping of space-time by mass, and the light rays observed from a star, galaxy, or other source are bent accordingly. In recent years, lensing has become an amazing way to test numerous astrophysical and cosmological theories. Strong

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lensing, or frameworks in which numerous pictures of a single source are recognizable or in which an Einstein ring is visible, can inform us about the Hubble constant and other cosmological parameters [19,20]. Factual estimations of lensing where the deflection of light is so weak to distinguish in a single background image, or weak lensing, gives an effective exploration of the matter distribution within the universe [21,22]. Weak lensing is an especially significant probe of dark matter [23,24] and has been considered as an instrument to distinguish general relativity from different theories [25,26]. In the literature [27–55], a lot of researchers have studied the deflection angle of light for different types of BHs and wormholes using the following the formula of Gibbons and Werner which was proposed using the Gauss-Bonnet theorem:

$$\tilde{\delta} = -\int\int_{\mathcal{A}_\infty} \tilde{K} dS, \tag{1}$$

where $\tilde{K}$ is the Gaussian curvature and $\mathcal{A}_\infty$ represents the infinite region of the surface. Recently, weak gravitational lensing by wormholes was studied and the deflection angle via naked singularities was calculated [56].

The main motivation of this paper is to study a conceivable extension of calculations of the bending angle of light for an Einstein-Maxwell-Dilaton-axion (EMDA) BH [57,58]. To do so, we calculate the deflection angle using the null geodesic technique and then look at a relation between the Gauss-Bonnet theorem and bending angle of light. After comparing these results, we investigate the deflection angle in a plasma medium for the given BH. The paper is organized as follows. In Sec. II we introduce the metric for an EMDA BH. In Sec. III we investigate the deflection angle using the geodesic method. Section IV contains the calculation of the deflection angle of light by an EMDA BH without a plasma medium in the weak-field approximations. Section V provides the deflection angle for an EMDA BH in the presence of a plasma medium. Finally, in Sec. VI we present our conclusion.

II. METRIC TENSOR OF A BLACK HOLE IN EINSTEIN-MAXWELL-DILATON-AXION THEORY

The EMDA black hole metric in spherical coordinates is defined as [58]

$$ds^2 = -\tilde{A}(r)dt^2 + \frac{dr^2}{\tilde{B}(r)} + \tilde{C}(r)[d\vartheta^2 + \sin^2\vartheta d\varphi^2], \tag{2}$$

where

$$\tilde{A}(r) = \tilde{B}(r) = \left(1 - \frac{2\tilde{M}}{r - 2r_0}\right) \quad \text{and,} \quad \tilde{C}(r) = (r - 2r_0)^2, \tag{3}$$

where $\tilde{M}$ represents the mass of the BH and r_0 is the parameter of the dilaton-axion field with $r_0 \approx a^2 e^{-a\varphi} Q^2/4m_\circ$, a is a constant free parameter, Q represents the charge of the BH, and $m_\circ \approx m + r_\circ$.

III. CALCULATION OF THE DEFLECTION ANGLE USING THE GEODESIC METHOD

In order to find the geodesics of an EMDA BH, the Lagrangian $\tilde{\mathcal{L}}$ can be given by the metric (2) as follows:

$$\begin{aligned}2\tilde{\mathcal{L}} = &-\left(1 - \frac{2\tilde{M}}{r(s) - 2r_0(s)}\right)\dot{t}^2(s) \\ &+ \left(1 - \frac{2\tilde{M}}{r(s) - 2r_0(s)}\right)^{-1}\dot{r}^2(s) \\ &+ (r(s) - 2r_0(s))^2(\dot{\vartheta}^2(s) + \sin^2\vartheta(s)\dot{\varphi}^2(s)).\end{aligned} \tag{4}$$

By considering $2\tilde{\mathcal{L}} = 0$ for photons, we have two constants of motion of the geodesics in the equatorial plane $\vartheta = \frac{\pi}{2}$,

$$\tilde{\mathcal{P}}_\varphi = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\varphi}} = 2(r(s) - 2r_0)^2\dot{\varphi}(s) = \tilde{\ell}, \tag{5}$$

$$\tilde{\mathcal{P}}_t = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{t}} = -2\left(1 - \frac{2\tilde{M}}{r(s) - 2r_0}\right)\dot{t}(s) = -\tilde{\epsilon}. \tag{6}$$

Furthermore, we consider a new variable $\xi(\varphi)$ which is related to the old radial coordinate as $r = \frac{1}{\xi(\varphi)}$, which leads to the identity

$$\frac{\dot{r}}{\dot{\varphi}} = \frac{dr}{d\varphi} = -\frac{1}{\xi^2}\frac{d\xi}{d\varphi}. \tag{7}$$

For the sake of simplicity, we use the metric conditions $\tilde{\epsilon} = 1$ & $\tilde{\ell} = b$ (note that b is the impact parameter) for Eqs. (4)–(7), and after some algebraic manipulations we have the following relation:

$$\frac{1}{\xi^4}\left(\frac{d\xi}{d\varphi}\right)^2\left(1 - \frac{2\tilde{M}}{1/\xi - 2r_0}\right)^{-1} - \frac{(1/\xi - 2r_0)^4}{b^2}\left(1 - \frac{2\tilde{M}}{1/\xi - 2r_0}\right)^{-1} + (1/\xi - 2r_0)^2 = 0. \tag{8}$$

The above equation implies

$$\left(\frac{d\varphi}{d\xi}\right) = \frac{\pm b}{\sqrt{1-2r_0\xi}} \frac{1}{\sqrt{1+(-6\xi+2b^2\xi^3)r_0+12r_0^2\xi^2-8r_0^3\xi^3-b^2(\xi^2-2\tilde{M}\xi^3)}}. \tag{9}$$

In order to derive the solution of Eq. (9), we use the following relation [59]:

$$\Delta\varphi = \pi + \tilde{\delta}, \tag{10}$$

where $\tilde{\delta}$ represents the deflection angle. The deflection angle can be obtained by following the same procedure as in Ref. [60],

$$\tilde{\delta} = 2|\varphi_{\xi=1/b} - \varphi_{\xi=0}| - \pi \tag{11}$$

and

$$\tilde{\delta} = \left| \int_0^{1/b} \left(\frac{d\varphi}{d\xi}\right) d\xi \right| - \pi = \left| \int_0^{1/b} \frac{b}{\sqrt{1-2r_0\xi}} \frac{1}{\sqrt{1+(-6\xi+2b^2\xi^3)r_0+12r_0^2\xi^2-8r_0^3\xi^3-b^2(\xi^2-2\tilde{M}\xi^3)}} \right| - \pi. \tag{12}$$

Here we neglect all r_0^2 and r_0^3 terms,

$$\tilde{\delta} = \left| \int_0^{1/b} \frac{b(1+r_0\xi)}{\sqrt{1+(-6\xi+2b^2\xi^3)r_0-b^2(\xi^2-2\tilde{M}\xi^3)}} d\xi \right| - \pi. \tag{13}$$

Then we proceed by introducing a new variable $y = 1/b$ and expand in a Taylor series around y. After we evaluate the integral for leading-order terms in $\tilde{M}$ and r_0, the deflection angle in the weak deflection limit approximation is found to be

$$\tilde{\delta} \simeq \frac{4\tilde{M}}{b} + \frac{3r_0\tilde{M}\pi}{2b^2}. \tag{14}$$

The deflection angle depends on the mass $\tilde{M}$, the parameter r_0, and the impact parameter b. Increasing the value of the dilaton-axion parameter r_0 increases the deflection angle in the weak-field limits.

IV. CALCULATION OF THE DEFLECTION ANGLE USING THE GAUSS-BONNET THEOREM

In order to find the null geodesics ($ds^2 = 0$), the metric can be written in the following simplified form [49]:

$$dt^2 = \frac{dr^2}{\tilde{A}(r)^2} + \frac{\tilde{C}(r)d\varphi^2}{\tilde{A}(r)}. \tag{15}$$

We consider a new coordinate r^* which satisfies the metric tensor $g_{\mu\nu}$ as

$$dt^2 = g_{\mu\nu}dx^\mu dx^\nu = dr^{*2} + f^2(r^*)d\varphi^2, \tag{16}$$

where

$$f(r^*) = \sqrt{\frac{\tilde{C}(r)}{\tilde{A}(r)}}. \tag{17}$$

In order to consider the Gauss-Bonnet theorem, we first calculate the Gaussian curvature $\tilde{K}$ of the optical spacetime which is an intrinsic property of spacetime. The Gaussian curvature can be calculated as [45]

$$\begin{aligned}\tilde{K} &= -\frac{R_{r\varphi r\varphi}}{\det g_{r\varphi}} \\ &= -\frac{1}{f(r^*)}\frac{d^2 f(r^*)}{dr^{*2}} \\ &= -2\frac{\tilde{M}}{r^3} + 3\frac{\tilde{M}^2}{r^4} + \left(-6\frac{\tilde{M}}{r^4} + 12\frac{\tilde{M}^2}{r^5}\right) r_0.\end{aligned} \tag{18}$$

The Gaussian curvature depends on the mass of the BH.

The Gauss-Bonnet theorem (GBT) with boundary $\partial\tilde{\mathcal{H}}_R = \tilde{\gamma}_h \cup \mathcal{C}_R$ can be defined as [28]

$$\iint_{\tilde{\mathcal{H}}_R} \tilde{K} d\tilde{\sigma} + \oint_{\partial\tilde{\mathcal{H}}_R} \tilde{\kappa} dt + \sum_i \tilde{\theta}_i = 2\pi\tilde{\chi}(\tilde{\mathcal{H}}_R), \tag{19}$$

where $\tilde{\kappa}$ denotes the geodesic curvature, while $\tilde{\theta}_i$ represents the exterior angle at the ith vertex and we consider the nonsingular domain $\tilde{\mathcal{H}}_R$ outside of the light ray with Euler characteristic $\tilde{\chi}(\tilde{\mathcal{H}}_R) = 1$ [45]. When we consider $\mathcal{R} \to \infty$, the two jump angles $(\tilde{\theta}_{\mathcal{O}}, \tilde{\theta}_{\mathcal{S}})$ yield $\frac{\pi}{2}$, and if we consider the total sum of the jump angles at observer $\mathcal{O}$ and source $\mathcal{S}$, we get $(\tilde{\theta}_{\mathcal{O}} + \tilde{\theta}_{\mathcal{S}}) \to \pi$ [28]. So, the GBT can be rewritten as follows:

$$\iint_{\tilde{\mathcal{H}}_R} \tilde{K} d\tilde{\sigma} + \oint_{\mathcal{C}_R} \tilde{\kappa} dt \overset{\mathcal{R}\to\infty}{=} \iint_{\tilde{\mathcal{H}}_\infty} \tilde{\mathcal{K}} d\tilde{\sigma} + \int_0^{\pi+\tilde{\alpha}} d\phi = \pi. \tag{20}$$

Now, in order to compute the geodesic curvature $\tilde{\kappa}$ we follow $\tilde{\kappa}(\tilde{\gamma}_h) = 0$, since $\tilde{\gamma}_h$ represents the geodesic. Therefore, we get

$$\tilde{\kappa}(C_R) = |\nabla_{\dot{C}_R} \dot{C}_R|, \tag{21}$$

in which we can choose $C_R \coloneqq r(\phi) = R = \text{const}$, where C_R represents the circle segment of coordinate radius R. Thus, the nonzero radial part can be calculated as

$$(\nabla_{\dot{C}_R} \dot{C}_R)^r = \dot{C}_R^\phi (\partial_\phi \dot{C}_R^r) + \tilde{\Gamma}^{r(\text{opt})}_{\phi\phi} (\dot{C}_R^\phi)^2, \tag{22}$$

where $\tilde{\Gamma}^{r(\text{opt})}_{\phi\phi}$ is the Christoffel symbol related to the optical geometry. It is clear that the first term in the above equation cancels the second term, as $(\dot{C}_R^\phi)^2 = 1/f^2(r^*)$ and $\tilde{\Gamma}^{r(\text{opt})}_{\phi\phi} = f(r^*)f'(r^*)$. When $R \to \infty$, the geodesic curvature $\tilde{\kappa}$ becomes

$$\lim_{R\to\infty} \tilde{\kappa}(C_R) = \lim_{R\to\infty} |\nabla_{\dot{C}_R} \dot{C}_R| \to \frac{1}{R}, \tag{23}$$

and dt becomes

$$\lim_{R\to\infty} dt = \lim_{R\to\infty} \left(\frac{\tilde{C}}{\tilde{A}}\right)^{1/2} d\phi \to (R - 2r_0) d\phi. \tag{24}$$

By combining Eqs. (23) and (24), we can get $\tilde{\kappa}(C_R) dt \approx R d\phi$.

Now we discuss the deflection angle in the weak-field limits. It is a well-known fact that in weak-field regions light rays follow approximately straight lines; therefore, we consider the condition of $r = \frac{b}{\sin\varphi}$ zero order. By using Eqs. (18), (20), and (24), the formula for the deflection angle is obtained as follows:

$$\tilde{\delta} = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \tilde{K} d\tilde{\sigma}. \tag{25}$$

After using the above relation, the deflection angle $\tilde{\delta}$ for the leading order of $\tilde{M}$ can be calculated as

$$\tilde{\delta} \simeq \frac{4\tilde{M}}{b} + \frac{3r_0 \tilde{M} \pi}{2b^2}. \tag{26}$$

The deflection angle depends on the mass $\tilde{M}$, the parameter r_0, and the impact parameter b. Increasing the value of the dilaton-axion parameter r_0 increases the deflection angle in the weak-field limits.

V. WEAK GRAVITATIONAL LENSING OF AN EMDA BH IN A PLASMA MEDIUM

In this section, we study the effect of a plasma medium on the weak gravitational lensing by an EMDA BH. The refractive index for the EMDA BH is given as [41]

$$\tilde{n}(r) = \sqrt{1 - \frac{\tilde{\omega}_e^2}{\tilde{\omega}_\infty^2}\left(1 - \frac{2\tilde{M}}{r - 2r_0}\right)}, \tag{27}$$

where $\tilde{\omega}_e$ and $\tilde{\omega}_\infty$ denote the electron and photon plasma frequencies, respectively.

In order to study the application of the Gauss-Bonnet theorem to the determination of the bending angle proposed by Gibbons and Werner [28], we consider a two-dimensional Riemannian manifold $(\mathcal{M}^{\text{opt}}, g^{\text{opt}}_{mn})$ with the optical metric $g^{\text{opt}}_{mn} = -\frac{n^2}{g_{00}} g_{mn}$. The corresponding optical metric is defined as follows:

$$d\tilde{\sigma}^2 = g^{\text{opt}}_{mn} dx^m dx^n = \frac{\tilde{n}^2(r)}{\tilde{A}(r)}\left(\frac{1}{\tilde{A}(r)} dr^2 + \tilde{C}(r) d\varphi^2\right),$$
$$\text{where } m, n = 1, 2, 3... \tag{28}$$

This metric preserves the angle between two curves at a given point and is conformally related to the metric (2), when we choose the spatial section $t = \text{const}$, $\theta = \pi/2$. By inserting Eq. (27) into Eq. (28), the optical metric takes the form

$$d\tilde{\sigma}^2 = \frac{r(\tilde{\omega}_\infty^2 - \tilde{\omega}_e^2) - 2r_0(\tilde{\omega}_\infty^2 - \tilde{\omega}_e^2) + 2\tilde{M}\tilde{\omega}_e^2}{(r - 2r_0 - 2\tilde{M})\tilde{\omega}_\infty^2} \times \left(\frac{dr^2}{1 - \frac{2\tilde{M}}{r - 2r_0}} + r^2 d\varphi^2\right). \tag{29}$$

The Gaussian curvature can be calculated as follows:

$$\tilde{K} = -\frac{R_{r\varphi r\varphi}(g^{\text{opt}})}{\det(g^{\text{opt}})} = \frac{\tilde{M}({\omega_e}^2 r - 2{\omega_\infty}^2 r + 4r_0{\omega_e}^2 - 6r_0{\omega_\infty}^2){\omega_\infty}^2}{({\omega_e}^2 - {\omega_\infty}^2)^2 r^4}. \tag{30}$$

Moreover, we have

$$\left.\frac{d\tilde{\sigma}}{d\varphi}\right|_{\mathcal{C}_R} = n(R)\left(\frac{\tilde{C}(R)}{\tilde{A}(R)}\right)^{1/2}, \tag{31}$$

which implies

$$\lim_{R\to\infty} \tilde{\kappa}_g \left.\frac{d\tilde{\sigma}}{d\varphi}\right|_{\mathcal{C}_R} \approx \frac{1}{R}. \tag{32}$$

By taking the straight-line approximation $r = \frac{b}{\sin\varphi}$ and for the limit $R \to \infty$, the Gauss Bonnet theorem takes the form

$$\lim_{R\to\infty} \int_0^{\pi+\tilde{\delta}} \left[\tilde{\kappa}_g \frac{d\tilde{\sigma}}{d\varphi}\right]\Bigg|_{C_{\mathcal{R}}} d\varphi = \pi - \int_0^{\pi} \int_{\frac{b}{\sin\varphi}}^{R} \tilde{K} d\tilde{\sigma}. \quad (33)$$

So, the deflection angle is obtained as

$$\tilde{\delta} = 3/2\frac{\tilde{M} r_0 \pi}{b^2} + 4\frac{\tilde{M}}{b} + 5/4\frac{{\omega_e}^2 \tilde{M} r_0 \pi}{{\omega_\infty}^2 b^2} + 4\frac{{\omega_e}^2 \tilde{M}}{b {\omega_\infty}^2} + \mathcal{O}(\tilde{M}^2). \quad (34)$$

The above equation indicates the photon rays' motion in a medium of homogeneous plasma. It is worth noting that if we neglect the plasma effects, i.e., $\frac{\omega_e^2}{\omega_\infty^2} \to 0$, then Eq. (34) reduces to Eq. (26), and thus we observe that the plasma effects can be removed. Moreover, the dilaton-axion parameter r_0 increases the deflection angle in a homogeneous plasma medium in the weak-field limits.

VI. CONCLUSION

In this work, we have studied the deflection angle for an EMDA BH. To do so, by considering the null geodesic method as well as new geometric techniques (Gauss-Bonnet theorem and optical geometry) established by Gibbons and Werner, we have calculated the deflection angle for an EMDA BH. It is also important to note that the deflection of a light ray is calculated outside of the lensing area, which shows that the gravitational lensing effect is a global and even topological effect, i.e., there is more than one light ray converging between the source and observer. Hence, the deflection angle of a photon is calculated as follows:

$$\tilde{\delta} \simeq \frac{4\tilde{M}}{b} + \frac{3 r_0 \tilde{M} \pi}{2b^2}. \quad (35)$$

On the other hand, the deflection angle of a photon is also obtained as

$$\tilde{\delta} = 3/2\frac{\tilde{M} r_0 \pi}{b^2} + 4\frac{\tilde{M}}{b} + 5/4\frac{{\omega_e}^2 \tilde{M} r_0 \pi}{{\omega_\infty}^2 b^2} + 4\frac{{\omega_e}^2 \tilde{M}}{b {\omega_\infty}^2} + \mathcal{O}(\tilde{M}^2). \quad (36)$$

Hence, we show that the dilaton-axion parameter r_0 increases the deflection angle in a homogeneous plasma medium in the weak-field limits.

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