Effect of the hair on deflection angle by asymptotically flat black holes in Einstein-Maxwell-dilaton theory

W. Javed,[1,*] J. Abbas,[1,†] and A. Övgün[2,3,‡]

[1]Department of Mathematics, University of Education, Township, Lahore-54590, Pakistan
[2]Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950 Valparaíso, Chile
[3]Physics Department, Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, North Cyprus 99628, via Mersin 10, Turkey

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In this paper, we are interested in a model of exact asymptotically flat charged hairy black holes in the background of a dilaton potential. We study the weak gravitational lensing in the spacetime of a hairy black hole in the Einstein-Maxwell theory with a nonminimally coupled dilaton and its nontrivial potential. In doing so, we use the optical geometry of the flat charged hairy black hole for some range of parameter γ. For this purpose, by using the Gauss-Bonnet theorem, we obtain the deflection angle of the photon in a spherically symmetric and asymptotically flat spacetime. Moreover, we also investigate the impact of the plasma medium on weak gravitational lensing by an asymptotically flat charged hairy black hole with a dilaton potential. Our analyses show the effect of the hair on the deflection angle in weak-field limits.

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I. INTRODUCTION

At the darkest points in the Universe, their boundaries perilous and invisible, space warps. The singularity constitutes the center of a black hole and is hidden by the object's surface, the event horizon. A black hole is a location in space that possesses so much gravity that nothing can escape its pull, even light. It is said that fact is sometimes stranger than fiction, and nowhere is this more true than in the case of black holes. Since the first image of a black hole by the Event Horizon Telescope [1], physicists now try to take even sharper images so that Einstein's theory of general relativity can be tested and also to see the properties of the black holes [2–10], because there are many theoretically obtained black hole solutions with different properties.

According to Einstein, a black hole (BH) can be described only in terms of its mass and spin, which is known as "no-hair" theorem. No hair means that information about the physical state of matter must be lost as the matter is sucked into a BH; otherwise, this information would distinguish one BH from another. In 1974, Hawking made the landmark conjecture that BHs do not simply suck in everything but rather behave as blackbodies that emit radiation as well as absorbing it [11]. He calculated the blackbody temperature of a BH known as the Hawking temperature. Having a distinct temperature implies that a BH has entropy, which Hawking also calculated. Entropy is a measure of the number of different ways the microscopic constituents of a BH can arrange themselves. This goes against the no-hair theorem, which says that a BH can be arranged only in one way as defined by its mass and spin. Recently, Hawking with his colleagues suggested that information-preserving massless particles known as soft hair could surround BHs. They have calculated the entropy of a BH that has a certain kind of soft hair, which leads to Hawking's original calculation of BH entropy [12].

The Einstein-Maxwell-dilaton (EMD) theory is an arousing and highly motivated theory to examine the effect of new essential degrees of freedom in order to prove the no-hair theorem because of the existence of hairy BH solutions in contact with the vector, scalar, and tensor spectrums. It is to be observed that, in the background of asymptotically anti–de Sitter (AdS) geometries, there must exist exact scalar-hairy BH solutions with respect to specific scalar potentials [13,14].

At the back of the no-hair conjecture, the simplest physical visualization is the following: After the gravitational collapse, the matter fields left over in the exterior region would finally be immersed by the BH itself or be radiated off to infinity. Although the hair of a BH is located at the outer side of event horizon, the question then arises of how there exists a possibility that the matter can hover in an immense gravitational field without collapsing entirely. Intuitively, the response to the question could be that this is possible if the internal pressure is large up to a sufficient degree. On the other hand, such a type of intuition does work only close to the event horizon, and it must be noted

*wajiha.javed@ue.edu.pk; wajihajaved84@yahoo.com
†jameelaabbas30@gmail.com
‡ali.ovgun@pucv.cl; https://www.aovgun.com

that the nonlinear interaction of matter fields provides the basis of the existence of hairy BHs [15].

A geometric spontaneous scalarization phenomenon has recently been examined [16–20]. It is noted that, in gravitational field and matter models, where a real or scalar field couples minimally to the curvature squared Gauss-Bonnet (GB) combination under particular options of the coupling function, there exist two possibilities of solutions. The standard vacuum BH solutions (bald) of general relativity (GR) and very recent "hairy" BH solutions with a scalar field characterization, both types of solutions deceiving the no-scalar hair conjecture. Moreover, it is suggested that the hairy BHs could form by means of spontaneous scalarization, since the standard vacuum BH solutions were shown to be perturbatively unstable. Following the previous analysis, Herdeiro and Radu [21,22] have examined the basics of the spontaneous scalarization phenomenon for BHs. They have elaborated this phenomenon within the quantum framework by giving two examples of spherically symmetric static asymptotically flat BHs in the effective field theory. One of the examples is that the trace anomaly occurs in the matter sector and provides a generalized form of the Reissner-Nordström BH solution incorporating F^4 correction, while, in the second case, it arises from the geometry sector and provides geometrically a noncommutative generalized Schwarzschild BH. In comparison, they have also discussed the scalarization phenomenon of Einstein-Maxwell-dilaton BHs. Because of a nonminimal coupling, they have investigated the connection of the scalarization mechanism with the quantum instabilities [23].

An important fundamental predication of GR is the phenomenon of gravitational lensing, which is the deflection of light in the presence of gravitational fields of compact objects. Solender (1801) proposed the gravitational lensing due to the Sun in the background of Newton's theory, while the absolute value of deflection angle $\delta = 4\, M_\odot/R_\odot \simeq 1.75$ arc sec was obtained in the background of GR by Chowlson (1929) and Einstein (1936). In depth, the study of gravitational lensing of compact objects initiated with the learning of facts of the double quasar $Q0957+561$ having redshift $\mathcal{Z} = 0.39$ (Young *et al.* 1980) [24–26]. They focused on the effects of the weak gravitational lensing, i.e., the lensing for which the deflection angle is very small, equal to a few arc seconds. Generally, there exists a differentiation between lensing on noncosmological as well as on cosmological distances. Moreover, the assumption of luminosity distances which are model dependent leads us to the opportunity to examine the mass distribution of dark matter halos, in particular, the inner density slopes [27–29]. Schunck, Fuchs, and Mielke [30] have investigated the deflection angle of a spherically symmetric halo built from a precisely solvable scalar field model incorporating Emden-type self-interaction. They have examined the gravitational lensing impact by considering a bosonic configuration and obtained the normalized projected mass as well as the corrections with respect to the pressure and analyze the weak-field dimensionless lens equation.

In this paper, we try to understand the effect of the hair on deflection angle by asymptotically flat black holes in the EMD theory, which is derived from a string theory at low-energy limits [31]. The EMD black holes have scalar hair, in addition to mass, rotation, and charge, so that they offer fascinating theoretical black hole models to check experimentally in black hole experiments and investigate possible differences between Einstein gravity and modified gravity theories [31–33].

Furthermore, let us now briefly review the Gauss-Bonnet theorem (GBT), which connects the topological surfaces. First, using Euler characteristic χ and a Riemannian metric g, one can choose the subset oriented surface domain as (D, χ, g) to find the Gaussian curvature K. Then the Gauss-Bonnet theorem is defined as follows [34]:

$$\int\int_D K\mathrm{d}S + \oint_{\partial D} \kappa \mathrm{d}t + \sum_i \theta_i = 2\pi\chi(D), \qquad (1)$$

where κ is the geodesic curvature for $\partial D: \{t\} \to D$ and θ_i is the exterior angle with the ith vertex. Following this approach, global symmetric lenses are considered to be Riemannian metric manifolds, which are geodesic spatial light rays. In optical geometry, we calculate the Gaussian optical curvature K to find the asymptotic bending angle which can be calculated as follows [34]:

$$\hat{\alpha} = -\int\int_{D_\infty} K\mathrm{d}S. \qquad (2)$$

Note that this equation is an exact result for the deflection angle. In this equation, we integrate over an infinite region of the surface D_∞ which is bounded by the light ray. By assumption, one can use the above relation only for asymptotically Euclidean optical metrics. Therefore, it will be interesting to see the form of the deflection angle in the case of nonasymptotically Euclidean metrics. This method has been applied in various papers for different types of spacetimes [35–63].

This paper is organized as follows: In Sec. II, we review some basic concepts about an asymptotically flat hairy BH. In Sec. III, we compute the Gaussian optical curvature for the deflection angle, calculate the deflection angle by using the GBT for $\gamma = 1$, and also find the deflection angle in a plasma medium. In Sec. IV, we calculate the deflection angle for $\gamma = \sqrt{3}$ and also find the deflection angle in a plasma medium. The last section comprises concluding remarks and results obtained from graphical analysis.

II. ASYMPTOTICALLY FLAT BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

In this section, we briefly review the asymptotically flat hairy BH in the EMD theory in the background of a dilaton potential. We consider the following action of the Einstein-Maxwell-dilaton theory [31,32]:

$$I[g_{\mu\nu},A_\mu,\phi]=\frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R-e^{\gamma\varphi}F^2-\frac{1}{2}(\partial\varphi)^2-V(\varphi)\right], \tag{3}$$

where

$$F^2=F_{\mu\nu}F^{\mu\nu}, \qquad (\partial\varphi)^2=\partial_\mu\varphi\partial^\mu\varphi, \tag{4}$$

$V(\varphi)$ is the dilaton potential, and $c=G_N=4\pi\epsilon_0=1$ by using the convention $k=8\pi$. The equations of motion for the gauge field, dilaton, and metric are, respectively, the following:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T^{\varphi}_{\mu\nu}+T^{\mathrm{EM}}, \tag{5}$$

$$\partial_\mu(\sqrt{-g}e^{\gamma\varphi}F^{\mu\nu})=0, \tag{6}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi)=\frac{dV(\varphi)}{d\varphi}+\gamma e^{\gamma\phi}F^2, \tag{7}$$

where the stress tensors of matter fields are defined as

$$T^{\varphi}_{\mu\nu}\equiv\frac{1}{2}\partial_\mu\phi\partial_\nu\varphi-\frac{1}{2}g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2+V(\phi)\right], \tag{8}$$

$$T^{\mathrm{EM}}_{\mu\nu}\equiv 2e^{\gamma\varphi}\left(F_{\mu\alpha}F^{\alpha}_{\nu}-\frac{1}{4}g_{\mu\nu}F^2\right). \tag{9}$$

For exact regular hairy BH solutions, for a general scalar potential there is a new method in a number of papers [33], by using a specific ansatz. For flat spacetime, we should apply the similar special ansatz for the metric and dilaton. For the sake of simplicity, we are going to focus on two particular cases for which the exponent coefficient of the dilaton coupling with the gauge field takes the values $\gamma=1$ and $\gamma=\sqrt{3}$, but for a more general solution there exists Ref. [32].

III. WEAK DEFLECTION ANGLE OF CALCULATION OF PHOTON LENSING FOR $\gamma=1$ BY GAUSS-BONNET THEOREM

For this solution, we consider the following scalar field potential:

$$V(\varphi)=2\alpha(2\varphi+\varphi\cosh\varphi-3\sinh\varphi), \tag{10}$$

where α is an arbitrary parameter. For $\gamma=1$, the static hairy BH metric and gauge field yields that

$$ds^2=\Omega(x)\left[-f(x)dt^2+\frac{\eta^2dx^2}{x^2f(x)}+d\Sigma^2\right],$$
$$F=\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu=\frac{qe^{-\phi}}{x}dt\wedge dx, \tag{11}$$

where η and q are defined as independent parameters of the solution and correspond to the mass and charge, respectively, of this BH. $d\Sigma^2=d\theta^2+\sin^2\theta d\phi^2$ is spherical line element, and the coordinate x is restricted to be positive: $x\in[0,\infty)$. We can suppose that $\eta>0$. One can use the conformal factor as follows:

$$\Omega(x)=\frac{x}{\eta^2(x-1)^2} \tag{12}$$

and then check that the equations of motion are satisfied for the following spacetime metric function:

$$f(x)=\alpha\left[\frac{x^2-1}{2x}-\ln(x)\right]+\frac{\eta^2(x-1)^2}{x}\left[1-\frac{2q^2(x-1)}{x}\right]. \tag{13}$$

It is appropriate to first find the black hole optical metric by imposing the null condition $ds^2=0$ and solving the spacetime metric for dt and also set the metric into equatorial plane with $\theta=\frac{\pi}{2}$, which yields

$$dt^2=\frac{\eta^2}{x^2f(x)^2}dx^2+\frac{1}{f(x)}d\varphi^2. \tag{14}$$

The optical geometry is in two dimensions and is obtained for a thermodynamically stable asymptotically flat hairy BH with a dilaton potential as follows [34]. By using the Gauss-Bonnet theorem, initially we find the Gaussian curvature $\mathcal{K}$ of the optical spacetime, as

$$\mathcal{K}=\frac{R_{\mathrm{icciScalar}}}{2}, \tag{15}$$

$$\begin{aligned}\mathcal{K}\approx&-\eta^2x-\frac{\eta^2}{x}+1/4\frac{\eta^2}{x^2}+3/2\eta^2+1/4x^2\eta^2\\&-1/2x\alpha\ln(x)-1/2\frac{\alpha\ln(x)}{x}+1/4\alpha x^2-1/4\frac{\alpha}{x^2}-x^2\eta^2q^2\\&+5\eta^2q^2x-11\frac{\eta^2q^2}{x^2}+16\frac{\eta^2q^2}{x}+3\frac{\eta^2q^2}{x^3}-12\eta^2q^2\\&+5\frac{\alpha q^2}{x}-3/2\frac{\alpha q^2}{x^3}-4\frac{\alpha q^2\ln(x)}{x^2}+3\frac{\alpha q^2\ln(x)}{x}\\&+x\alpha\ln(x)q^2-3\alpha q^2-1/2x^2\alpha q^2\\&+1/2x\alpha q^2-1/2\frac{\alpha q^2}{x^2}.\end{aligned} \tag{16}$$

For multiple images, we use the global theory (Gauss-Bonnet theorem) to relate with the local feature of the spacetime such that there is Gaussian optical curvature.

The above equation will be apply to calculate the deflection angle by taking a nonsingular domain S_R outside of the light ray (along with boundaries $\partial S_R = \gamma_g \cup C_R$) with Euler characteristic $\chi(S_R)$, Gaussian curvature $\mathcal{K}$, geodesic curvature k, and exterior jump angles $\alpha_i = (\alpha_O, \alpha_S)$ at the vertices:

$$\int\int_{S_R} \mathcal{K} dS + \oint_{\partial S_R} k dt + \sum_j \alpha_j = 2\pi\chi(S_R), \quad (17)$$

at a weak limit approximation $(\rho \to \infty)$, $\alpha_O + \alpha_S \to \pi$. Then the GBT becomes

$$\int\int_{S_R} \mathcal{K} dS + \oint_{C_r} k dt =^{\rho\to\infty} \int\int_{S_\infty} k dS + \int_0^{\pi+\Theta} d\varphi = \pi. \quad (18)$$

Now, by the geodesic property, the geodesic curvature vanishes, $k(\gamma_g) = 0$, and we get

$$k(C_R) = |\nabla_{\dot{C}_r \dot{C}_r}|, \quad (19)$$

with $C_r \coloneqq \rho(\varphi) = r = \text{const}$. Then the GBT reduces

$$\lim_{R\to\infty} \int_0^{\pi+\Theta} \left[k_g \frac{d\sigma}{d\varphi}\right]\bigg|_{C_R} d\varphi = \pi - \lim_{R\to\infty} \int\int_{S_R} \mathcal{K} dS. \quad (20)$$

Now, for radial distance,

$$k(C_R) dt = d\varphi. \quad (21)$$

Therefore,

$$\lim_{R\to\infty} k_g \frac{d\sigma}{d\varphi}\bigg|_{C_R} = 1. \quad (22)$$

In the weak-field regions, the light ray follows a straight line approximation, so that we can use the condition of $r = b/\sin\varphi$ at zero order:

$$\Theta = -\lim_{R\to\infty} \int_0^\pi \int_{b/\sin\varphi}^R \mathcal{K} dS, \quad (23)$$

where

$$KdS \approx 1/4 \frac{\eta^2}{x^2} + 1/4 \frac{\alpha}{x^2} - \frac{\eta^2 q^2}{x^2} - 1/2 \frac{\alpha q^2}{x^2}. \quad (24)$$

After simplification, we find the deflection angle of a photon for an asymptotically flat hairy BH in leading-order terms as

$$\tilde{\alpha} \simeq -1/2 \frac{\eta^2}{b} - 1/2 \frac{\alpha}{b} + 2 \frac{\eta^2 q^2}{b} + \frac{\alpha q^2}{b}. \quad (25)$$

Therefore, we can say that the GBT provides a global as well as a topological effect; this method is very useful for a quantitative tool and can be applied in any asymptotically flat metrics.

A. Photon lensing in a plasma medium

In this section, we analyze the effect of a plasma medium on the photon lensing by an asymptotically hairy black hole. The refractive index for a hairy black hole is as follows [35]:

$$n(x) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2}\left(\frac{x f(x)}{\eta^2 (x-1)^2}\right)}, \quad (26)$$

then, the corresponding optical metric yields that

$$d\tilde{\sigma}^2 = g_{jk}^{\text{opt}} dx^j dx^k = \frac{n^2(x)}{f(x)}\left(\frac{\eta^2}{x^2 f(x)} dx^2 + d\varphi^2\right). \quad (27)$$

The determinant of the above optical metric is

$$\det g_{x\varphi}^{\text{opt}} = \frac{x f(x)\omega_e^4 - 2\eta^2 \omega_\infty^2 \omega_e^2 (x-1)^2}{\eta^2 \omega_\infty^4 x f(x)^2 (x-1)^4}, \quad (28)$$

where the metric function is given as

$$f(x) = \alpha\left[\frac{x^2-1}{2x} - \ln(x)\right] + \frac{\eta^2 (x-1)^2}{x}\left[1 - \frac{2q^2 (x-1)}{x}\right]. \quad (29)$$

Now, we have

$$\frac{d\tilde{\sigma}}{d\varphi} = n(x)\left(\frac{\alpha^2 x^2}{f(x)}\right)^{1/2}, \quad (30)$$

hence, we get differently which goes to α:

$$\lim_{x\to\infty} k_g \frac{d\tilde{\sigma}}{d\varphi}\bigg|_{C_R} = 1. \quad (31)$$

We use straight line approximation $r = b/\sin\varphi$, for the limit $x \to \infty$, and then the GBT stated as

$$\lim_{x\to\infty} \int_0^{\pi+\Theta} \left[k_g \frac{d\tilde{\sigma}}{d\varphi}\right]\bigg|_{C_R} d\varphi = \pi - \lim_{x\to\infty} \int_0^\pi \int_{b/\sin\varphi}^x \mathcal{K} dS, \quad (32)$$

where

$$\mathcal{K}dS = -3/2\frac{{\omega_e}^2\eta^2q^2}{x^2{\omega_\infty}^2} - 3/2\frac{{\omega_e}^2\alpha q^2}{x^2{\omega_\infty}^2} + 1/4\frac{{\omega_e}^2\eta^2}{x^2{\omega_\infty}^2} + 3/8\frac{{\omega_\infty}^2\alpha}{x^2{\omega_\infty}^2} + 1/4\frac{\alpha}{x^2} + 1/4\frac{\eta^2}{x^2} - 1/2\frac{\alpha q^2}{x^2} - \frac{\eta^2q^2}{x^2}. \quad (33)$$

Note that we consider only the first-order terms. After simplification, we obtain the deflection angle in weak-field limits as follows:

$$\tilde{\alpha} \simeq 3\frac{{\omega_e}^2\eta^2q^2}{b{\omega_\infty}^2} + 2\frac{\eta^2q^2}{b} + 3\frac{{\omega_e}^2\alpha q^2}{b{\omega_\infty}^2} + \frac{\alpha q^2}{b} - 1/2\frac{{\omega_e}^2\eta^2}{b{\omega_\infty}^2} - 1/2\frac{\eta^2}{b} - 3/4\frac{{\omega_e}^2\alpha}{b{\omega_\infty}^2} - 1/2\frac{\alpha}{b}. \quad (34)$$

The above results show that the photon rays are moving in a medium of homogeneous plasma.

IV. NEW ASYMPTOTICALLY FLAT BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

In this section, we analyzed the exact asymptotically flat charged hairy BHs in the background of a dilaton potential. There exists literature related to this BH recently discussed by Anabalon, Astefanesei, and Mann [32]. We are interested in an action of the Einstein-Maxwell-dilaton theory ($\kappa = 8\pi G_N$):

$$I[g_{\mu\nu}, A_\mu, \phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}e^{\gamma\phi}F^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \quad (35)$$

where the gauge coupling and potential are functions of the dilaton. The corresponding equations of motion are

$$\nabla_\mu(e^{\gamma\phi}F^{\mu\nu}) = 0, \quad (36)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - \frac{\partial V}{\partial\phi} - \frac{1}{4}\gamma e^{\gamma\phi}F^2 = 0, \quad (37)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}[T^\phi_{\mu\nu} + T^{\mathrm{EM}}_{\mu\nu}], \quad (38)$$

where $T^\phi_{\mu\nu}$ and $T^{\mathrm{EM}}_{\mu\nu}$ are the stress tensors of the matter fields and represented as follows:

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right],$$
$$T^{\mathrm{EM}}_{\mu\nu} = e^{\gamma\phi}\left(F_{\mu\alpha}F_\nu^{\cdot\alpha} - \frac{1}{4}g_{\mu\nu}F^2\right). \quad (39)$$

Now, we consider only asymptotically flat solutions. To implement this condition, we require

$$\lim_{x\to 1}\Omega(x)f(x) = 1, \quad (40)$$

and, on this account, we fix f_0 to obtain [32]

$$f(x) = \frac{\eta^2}{\nu}\left(x^2 + \frac{2x^{2-\nu}}{\nu-2} - \frac{\nu}{\nu-2}\right) + f_1\left(\frac{x^{\nu+2}}{\nu+2} - x^2 + \frac{x^{2-\nu}}{2-\nu} + \frac{\nu^2}{\nu^2-4}\right) + \frac{Q^2\eta^2}{(1-p)\nu^2}\left(\frac{x^{3-p+\nu}}{3-p+\nu} + \frac{x^{3-p-\nu}}{3-p-\nu} - 2\frac{x^{3-p}}{3-p} - \frac{2\nu^2}{(3-p)(3-p+\nu)(3-p-\nu)}\right) + \frac{P^2\eta^4}{(1+p)\nu^2}\left(\frac{x^{3+p+\nu}}{3+p+\nu} + \frac{x^{3+p-\nu}}{3+p-\nu} - 2\frac{x^{3+p}}{3+p} - \frac{2\nu^2}{(3+p)(3+p+\nu)(3+p-\nu)}\right). \quad (41)$$

A. Solutions with a nontrivial dilaton potential ($\gamma = 1$)

For this solution, we consider the following scalar field potential [32]:

$$\Omega(x) = \frac{x}{\eta^2(x-1)^2}, \qquad \phi(x) = \ln(x), \quad (42)$$

$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2dx^2}{x^2f(x)} + d\theta^2 + \sin^2\theta d\varphi^2\right]. \quad (43)$$

Here we study only the $\gamma = 1$ case, which is smoothly connected with a solution of $\mathcal{N} = 4$ supergravity which gives the solutions of the metric function as follows:

$$A = \frac{Q}{x}dt + P\cos\theta d\varphi, \quad (44)$$

$$V(\phi) = \alpha[2\phi + \phi\cosh(\phi) - 3\sinh(\phi)], \quad (45)$$

$$f(x) = \frac{\eta^2(x-1)^2}{x} + \left[\frac{x}{4} - \frac{1}{4x} - \frac{1}{2}\ln(x)\right]\alpha + \frac{\eta^2(x-1)^3}{2x}(\eta^2P^2 - x^{-1}Q^2). \quad (46)$$

Note that the dilaton field is vanishing at the boundary $x = 1$ and also the dilaton potential is vanishing when $\alpha = 0$. At the boundary $x = 1$, one can find an asymptotically flat spacetime. After we make a change of coordinates using

$$\Omega(x) = r^2 + O(r^{-4}), \tag{47}$$

which is given for the $x < 1$ black holes by

$$x = 1 - \frac{1}{\eta r} + \frac{1}{2\eta^2 r^2} - \frac{1}{8\eta^3 r^3} + \frac{1}{2^7 \eta^5 r^5}, \tag{48}$$

the spacetime metric becomes asymptotically flat [32]:

$$g_{tt} = \Omega(x) f(x) = 1 - \frac{\alpha + 6\eta^2(\eta^2 P^2 - Q^2)}{12\eta^3 r} + O(r^{-2}), \tag{49}$$

$$g_{rr}^{-1} = \frac{x^2 f(x)}{\eta^2 \Omega(x)} \left(\frac{dr}{dx}\right)^2 = 1 - \frac{\alpha + 6\eta^2(\eta^2 P^2 - Q^2)}{12\eta^3 r} + O(r^{-2}). \tag{50}$$

It is noted that the scalar field potential is regular everywhere, except at the spacetime singularities.

V. DEFLECTION ANGLE OF PHOTONS BY ASYMPTOTICALLY FLAT HAIRY BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

Initially, we find the Gaussian curvature $\mathcal{K}$ of the optical spacetime, as

$$\mathcal{K} = \frac{R_{\text{icciScalar}}}{2} \tag{51}$$

and

$$\mathcal{K} = \frac{(-6\eta^4 p^2 + 6Q^2\eta^2 - \alpha)(-6\eta^4 p^2 + 6Q^2\eta^2 + 16\eta^3 r - \alpha)}{192\eta^6 r^4}. \tag{52}$$

In weak-field limits,

$$\mathcal{K} = -\frac{\alpha}{12\eta^3 r^3} + \frac{\alpha^2}{192\eta^6 r^4} + \frac{(-8\eta^3 r + \alpha)p^2}{16\eta^2 r^4} - \frac{(-8\eta^3 r + \alpha)Q^2}{16\eta^4 r^4} + \mathcal{O}(Q^3, p^3). \tag{53}$$

For multiple images, we use the global theory (Gauss-Bonnet theorem) to relate with the local feature of the spacetime such that there is Gaussian optical curvature.

In the weak-field regions, the light ray follows a straight line approximation, so that we can use the condition of $r = b/\sin\phi$ at zero order:

$$\tilde{\alpha} = -\lim_{R\to\infty} \int_0^{\pi} \int_{b/\sin\varphi}^{R} \mathcal{K} dS. \tag{54}$$

Now, by using Eq. (18), the deflection angle of a photon by an exact asymptotically flat charged hairy black hole with a dilaton potential in the weak-field limit is found as

$$\tilde{\alpha} = \frac{3Q^2 p^2 \pi}{32b^2} + \frac{\eta p^2}{b} - \frac{Q^2}{b\eta} + \frac{\pi Q^2 \alpha}{64 b^2 \eta^4} + \frac{\alpha}{6b\eta^3} + \mathcal{O}(Q^3, p^3). \tag{55}$$

A. Deflection angle of photons in plasma medium by asymptotically flat hairy black holes in Einstein-Maxwell-dilaton theory

In this section, we analyze the effect of a plasma medium on the photon lensing by an asymptotically hairy black hole. The refractive index for hairy black hole is as follows [35]:

$$n(x) = \sqrt{1 - \frac{\omega_e{}^2}{\omega_\infty{}^2}\left(\frac{x f(x)}{\eta^2 (x-1)^2}\right)}, \tag{56}$$

then, the corresponding optical metric yields that

$$d\tilde{\sigma}^2 = g_{jk}^{\text{opt}} dx^j dx^k = \frac{n^2(x)}{f(x)}\left(\frac{\eta^2}{x^2 f(x)} dx^2 + d\varphi^2\right). \tag{57}$$

The determinant of the above optical metric is

$$\det g_{x\varphi}^{\text{opt}} = \frac{x f(x)\omega_e{}^4 - 2\eta^2 \omega_\infty{}^2 \omega_e{}^2 (x-1)^2}{\eta^2 \omega_\infty{}^4 x f(x)^2 (x-1)^4}. \tag{58}$$

Now, by using Eqs. (17) and (29), in the weak-field limit the Gaussian optical curvature is stated as follows:

$$\mathcal{K} = -\frac{\omega_\infty{}^2(\omega_e{}^2 - 2\omega_\infty{}^2)Q^2}{4r^3\eta(\omega_e{}^2 - \omega_\infty{}^2)^2} + \frac{\omega_\infty{}^2(3Q^2\omega_e{}^2\omega_\infty{}^2 + 3Q^2\omega_\infty{}^4 + 2\eta r\omega_e{}^4 - 6\eta r\omega_e{}^2\omega_\infty{}^2 + 4\eta r\omega_\infty{}^4)p^2}{8r^4(\omega_e{}^2 - \omega_\infty{}^2)^3} + \frac{\alpha(\Psi)\omega_\infty{}^2}{48\eta^4 r^5(\omega_e{}^2 - \omega_\infty{}^2)^4}, \tag{59}$$

where

$$\Psi = 9Q^2\eta p^2\omega_e{}^4\omega_\infty{}^2 + 9Q^2\eta p^2\omega_e{}^2\omega_\infty{}^4 - 3\eta^2 p^2 r\omega_e{}^4\omega_\infty{}^2 + 3\eta^2 p^2 r\omega_\infty{}^6 + 3Q^2 r\omega_e{}^4\omega_\infty{}^2 - 3Q^2 r\omega_\infty{}^6 + 2\eta r^2\omega_e{}^6 - 8\eta r^2\omega_e{}^4\omega_\infty{}^2 + 10\eta r^2\omega_e{}^2\omega_\infty{}^4 - 4\eta r^2\omega_\infty{}^6. \tag{60}$$

Now, we have

$$\frac{d\tilde{\sigma}}{d\varphi} = n(x)\left(\frac{\alpha^2 x^2}{f(x)}\right)^{1/2}, \tag{61}$$

hence, we get differently which goes to α:

$$\lim_{x\to\infty} k_g \frac{d\tilde{\sigma}}{d\varphi}\bigg|_{C_R} = \alpha. \tag{62}$$

We use straight line approximation $r = b/\sin\varphi$, for the limit $x \to \infty$, and then the GBT is stated as

$$\lim_{x\to\infty}\int_0^{\pi+\tilde{\alpha}}\left[k_g\frac{d\tilde{\sigma}}{d\varphi}\right]\bigg|_{C_R} d\varphi = \pi - \lim_{x\to\infty}\int_0^{\pi}\int_{b/\sin\varphi}^{x}\mathcal{K}dS. \tag{63}$$

After simplification, we obtain

$$\begin{aligned}\tilde{\alpha} =& \frac{21\pi Q^2p^2{\omega_e}^2}{64{\omega_\infty}^2b^2} + \frac{3Q^2p^2\pi}{32b^2} + \frac{\eta p^2{\omega_e}^2}{{\omega_\infty}^2 b} + \frac{\eta p^2}{b} \\ &- \frac{7\pi\alpha p^2{\omega_e}^2}{128{\omega_\infty}^2b^2\eta^2} - \frac{\pi\alpha p^2}{64b^2\eta^2} - \frac{Q^2{\omega_e}^2}{{\omega_\infty}^2 b\eta} \\ &- \frac{Q^2}{b\eta} + \frac{7\pi Q^2\alpha{\omega_e}^2}{128{\omega_\infty}^2\eta^4b^2} + \frac{\pi Q^2\alpha}{64b^2\eta^4} + 1/6\frac{\alpha{\omega_e}^2}{{\omega_\infty}^2\eta^3 b} + 1/6\frac{\alpha}{b\eta^3}.\end{aligned} \tag{64}$$

The proposed deflection angle shows that the photon rays are moving in a medium of homogeneous plasma.

VI. CONCLUSION

In this paper, we obtain the deflection angle of a photon to the spherically symmetric and asymptotically flat spacetime of a hairy BH with an Einstein-Maxwell-dilaton system in the weak-field limit. To this end, we set the photon rays on the equatorial plane in the black hole spacetime. For this purpose, we have used the GBT and obtain the deflection angle of the photon by integrating a domain outside the impact parameter. Moreover, we also found the deflection angle of a photon by an asymptotically flat hairy BH in a plasma medium. We examined that the proposed deflection angle shows that gravitational lensing can be affected from the hair of the black hole, and it is a global effect as well as a valuable tool to study the nature of singularities of black holes.

In our analysis, we obtain the deflection angle of a photon to the spherically symmetric and asymptotically flat spacetime of a hairy BH with an EMD system in the weak-field limit by using the Gauss-Bonnet theorem. The computed deflection angle defined by (55) is stated as follows:

$$\tilde{\alpha} = \frac{3Q^2p^2\pi}{32b^2} + \frac{\eta p^2}{b} - \frac{Q^2}{b\eta} + \frac{\pi Q^2\alpha}{64b^2\eta^4} + \frac{\alpha}{6b\eta^3} + \mathcal{O}(Q^3, p^3). \tag{65}$$

It is noted that, for the selection of mass term $\eta = m$, $p = 2$, in the absence of charge $Q = 0$, and $\alpha = 0$, the above equation reduces up to the first-order term of the deflection angle for a Schwarzschild black hole [34]:

$$\tilde{\alpha} \simeq \frac{4m}{b}, \tag{66}$$

where m is the black hole mass. The significance of the obtained result in the weak-field approximation is that the deflection of a light ray is evaluated by taking a domain outside of the lensing region, which implies that the impact of the gravitational lensing is a global effect in such a way that there are multiple light rays converging between the source and observer.

Furthermore, by considering the homogeneous plasma medium, we evaluate the deflection angle of a photon given by (64) for an asymptotically flat hairy BH, which is given below:

$$\begin{aligned}\tilde{\alpha} =& \frac{21\pi Q^2p^2{\omega_e}^2}{64{\omega_\infty}^2b^2} + \frac{3Q^2p^2\pi}{32b^2} + \frac{\eta p^2{\omega_e}^2}{{\omega_\infty}^2 b} + \frac{\eta p^2}{b} - \frac{7\pi\alpha p^2{\omega_e}^2}{128{\omega_\infty}^2b^2\eta^2} \\ &- \frac{\pi\alpha p^2}{64b^2\eta^2} - \frac{Q^2{\omega_e}^2}{{\omega_\infty}^2 b\eta} - \frac{Q^2}{b\eta} + \frac{7\pi Q^2\alpha{\omega_e}^2}{128{\omega_\infty}^2\eta^4b^2} + \frac{\pi Q^2\alpha}{64b^2\eta^4} \\ &+ 1/6\frac{\alpha{\omega_e}^2}{{\omega_\infty}^2\eta^3 b} + 1/6\frac{\alpha}{b\eta^3}.\end{aligned} \tag{67}$$

The above equation can be expressed as

$$\begin{aligned}\tilde{\alpha} =& \left[\frac{\eta p^2}{b} + \frac{3Q^2p^2\pi}{32b^2} - \frac{\alpha\pi p^2}{64b^2\eta^2} - \frac{Q^2}{b\eta} + \frac{\alpha Q^2\pi}{64b^2\eta^4} + \frac{\alpha}{6b\eta^3}\right] \\ &+ \left(\frac{\omega_e}{\omega_\infty}\right)^2\left[\frac{21\pi Q^2p^2}{64b^2} + \frac{\eta p^2}{b} - \frac{7\pi\alpha p^2}{128b^2\eta^2} \right. \\ &\left. - \frac{Q^2}{b\eta} + \frac{7\pi Q^2\alpha}{128\eta^4b^2} + \frac{\alpha}{6\eta^3 b}\right].\end{aligned} \tag{68}$$

For $Q = 0$ and $p = 2$, one can obtain the following form:

$$\begin{aligned}\tilde{\alpha} =& \left[\frac{4\eta}{b} - \frac{\alpha\pi}{16\eta^2b^2} + \frac{\alpha}{6b\eta^3}\right] + \left(\frac{\omega_e}{\omega_\infty}\right)^2\left[\frac{4\eta}{b} - \frac{7\pi\alpha}{32b^2\eta^2} + \frac{\alpha}{6\eta^3 b}\right], \\ \tilde{\alpha} =& \frac{4\eta}{b}\left[1 + \left(\frac{\omega_e}{\omega_\infty}\right)^2\right] + \mathcal{O}(\eta^2).\end{aligned} \tag{69}$$

Because of the presence of the plasma medium, the gravitational deflection angle increases and depends upon the frequency of photons. In a homogeneous plasma medium, the photons have a smaller frequency or greater wavelengths and are deflected by a larger angle near the gravitating center. For $\omega \to \omega_e$, the influential difference in the gravitational deflection angles is substantial for longer wavelengths, which is possible only for radio waves. In this regard, the gravitational lens in a plasma acts as a radio spectrometer [64]. Crisnejo and Gallo [35] have studied the dynamics of light rays in a cold nonmagnetized plasma medium. For this purpose, they have obtained the deflection angle for Schwarzschild spacetime in a homogenous plasma medium by using the Gauss-Bonnet theorem, i.e.,

$$\tilde{\alpha} = \frac{2m}{b}\left(1 + \frac{1}{1-(\omega_e/\omega_\infty)^2}\right) + \mathcal{O}(m^2). \quad (70)$$

In comparison with the deflection angle obtained in Ref. [35], with the choice of parameters $\eta = m$, $p = 2$, $\alpha = 0$, and $Q = 0$, our proposed deflection angle (67) approximates the deflection angle of the Schwarzschild black hole (70). In the future, astrophysical observations might shed light on the effect of hair on the deflection angle. Any discovery of hair would be an important signal beyond general relativity.

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