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Charged fermions tunneling from stationary axially symmetric black holes with generalized uncertainty principle

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In this paper, we study the tunneling of charged fermions from the stationary axially symmetric black holes using the generalized uncertainty principle (GUP) via Wentzel, Kramers, and Brillouin (WKB) method. The emission rate of the charged fermions and corresponding modified Hawking temperature of Kerr–Newman black hole, Einstein–Maxwell-dilaton-axion (EMDA) black hole, Kaluza–Klein dilaton black hole, and then, charged rotating black string are obtained and we show that the corrected thermal spectrum is not purely thermal because of the minimal scale length which cause the black hole's remnant.

Keywords: Black holes; Hawking radiation; black hole thermodynamics; black hole temperature; modified Dirac equation; Hamilton–Jacobi method; WKB approximation.

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1. Introduction

According to general relativity, black holes are such dense objects in the universe that not even light can escape. The classical Einstein equations state that black holes are disturbingly simple; their only properties are mass, electrical charge and angular momentum. The simplest solution of the Einstein equations in general spherically symmetric vacuum is a Schwarzschild solution, which depends only on a single parameter of mass. In addition, the Reissner–Nordström solution, which has two parameters of electric charge and mass, is the general spherically symmetric solution to the Einstein–Maxwell (EM) equations. On the other hand, the axially symmetric solution of EM equations is Kerr–Newman solution which depends on three parameters; mass M, electric charge Q, and angular momentum J. Furthermore, one can use scalar fields together with string theory to solve mysteries of the dark matter and dark energy. The Einstein–Maxwell-dilaton-axion (EMDA) gravity is the low energy limit of the bosonic sector of the heterotic string theory.[1] EMDA gravity model is a generalization of EM gravity which contains the dilaton and the axion scalar fields. Moreover, black holes are interpreted microscopically in string theory as bound states of explicitly specified constituents. For example: the original Kaluza–Klein theory in four dimensions, obtained by compactification of 5-dimensional pure gravity on a circle which contains the following fields: a U(1) gauge field, a scalar field, and gravity.[2,3]

In 1974, Hawking predicted that black holes would release black body radiation, known as Hawking Radiation.[4,5] Moreover, this effect cause the information loss paradox, because of the thermal nature of the radiation. Nowadays, there have been many works on the derivation of the Hawking temperature using various techniques and methods.[6–8] One of the famous method is the quantum tunneling method firstly used by Krauss–Wilczek–Parkih and also semiclassical method of tunneling using the Hamilton–Jacobi approach.[9–60] Recently, the effect of the quantum gravity has been investigated using the generalized uncertainty principle (GUP) in different spacetimes, which indicate that the rate of the tunneling of particles deviates from pure thermality and satisfy the unitary theory. Furthermore, the researches on string theory, loop quantum gravity, double special relativity show that there is a possibility to existences of the minimal observable length which is the main ingredient of the GUP.[61–66] Briefly, this modification on the uncertainty principle is as follows[67]:

$$[x_i, p_i] = i\hbar\delta_{ij}[1 + \beta p^2],$$

where

$$x_i = x_{0i}, \quad p_i = p_{0i}(1 + \beta p^2),$$
$$p^2 = p_i p^i \simeq -\hbar[\partial_i\partial^i - 2\beta\hbar^2(\partial_j\partial^j)(\partial^i\partial_i)].$$

Note that x_{0i} and p_{0i} satisfy the canonical commutation relations $[x_{0i}; p_{0j}] = i\hbar\delta_{ij}$ and $\beta = \beta_0 l_p^2/\hbar^2$, β_0 are dimensionless parameters with the Planck length l_p. Using these commutation relations, the GUP can be written as[67]

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \beta(\Delta p)^2].$$

In recent years, there have been many publications including the effect of GUP.[68–87] Moreover, recently, tunneling of the uncharged particles from rotating black holes has been studied,[88] however, there is no completely agreement with the literature. The main aim of the paper is to obtain correct Hawking temperature using the tunneling of fermions from the stationary axially symmetric black holes. The idea is to get the quantum signature of the correlated Hawking quanta as a proof of the Hawking effect and to acquire the GUP effects on Hawking temperature, we study the general stationary axially symmetric black holes.

The paper is organized as follows. In Sec. 2, we briefly review the method of tunneling using the charged fermions from the stationary axially symmetric black holes via GUP. In Sec. 3, the modified Hawking temperature of Kerr–Newman black hole is obtained. In Sec. 4, we study the temperature of EMDA black hole. In Sec. 5, we calculate the modified temperature of Kaluza–Klein dilaton black hole. Then, in Sec. 6, we obtain the effect of the GUP on the charged rotating black string. In Sec. 7, we summarize our results.

2. Modified Temperature for General Stationary Axially Symmetric Black Hole

In this section, we develop a general method to study charged fermions tunneling from stationary axially symmetric black holes with GUP. We consider general 4-dimensional line element for stationary axially symmetric black holes and discuss tunneling with general line element. Then we calculate the tunneling probability and we give general formula for modified Hawking temperature. The general line element and electromagnetic potential of axially symmetric black hole can be written as

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\xi\xi}\,d\xi^2 + g_{\phi\phi}\,d\phi^2 + 2g_{t\phi}\,dt\,d\phi, \tag{1}$$

with

$$A_\mu = A_t\,dt + A_\phi\,d\phi. \tag{2}$$

The line element can be transformed into the following form:

$$ds^2 = \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right)dt^2 + g_{rr}\,dr^2 + g_{\xi\xi}\,d\xi^2 + g_{\phi\phi}\left(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}}dt\right)^2. \tag{3}$$

The angular velocity of the black hole with line element (1) is defined as

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}. \tag{4}$$

It is a well-known fact that the tunneling probability of particle is independent from the coordinates system, so for simplicity of our calculation, we perform dragged coordinate transformation. After transformation of coordinates as $d\phi = \Omega\, dt$, the dragged metric can be written as

$$d\hat{s}^2 = \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right) dt^2 + g_{rr}\, dr^2 + g_{\xi\xi}\, d\xi^2,$$
$$d\hat{s}^2 \equiv -F\, dt^2 + \frac{dr^2}{G} + H^2\, d\xi^2. \tag{5}$$

The transformed metric represents 3-dimensional hyperspace in the 4-dimensional spacetime. The corresponding electromagnetic vector potential is

$$\hat{A}_i = \hat{A}_t\, dt = (A_t + \Omega A_\phi) dt. \tag{6}$$

Here quantities are 3-dimensional quantities. To discuss the charged fermions tunneling with GUP, we use modified Dirac equation. The modified Dirac equation for the fermions field of mass m and charge e can be written as[89]

$$\Big[i\gamma^0\partial_0 + i\gamma^i\partial_i(1-\beta m^2) + i\gamma^i\beta\hbar^2(\partial_j\partial^j)\partial_i + \frac{m}{\hbar}(1+\beta\hbar^2\partial_j\partial^j - \beta m^2)$$
$$- \gamma^\mu \frac{e}{\hbar}\hat{A}_\mu(1+\beta\hbar^2\partial_j\partial^j - \beta m^2) + i\gamma^\mu\Omega_\mu(1+\beta\hbar^2\partial_j\partial^j - \beta m^2)\Big]\Psi = 0, \tag{7}$$

where $\Omega_\mu = \frac{1}{2}i\Gamma_\mu^{\nu\lambda}\Sigma_{\nu\lambda}$, $\Sigma_{\mu\nu} = \frac{1}{4}i[\gamma^\mu, \gamma^\nu]$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$, with i, $j = 1, 2$ and μ, ν, $\lambda = 0, 1, 2$. For transformed metric (5) gamma matrices γ^μ can be constructed as

$$\gamma^t = \frac{1}{\sqrt{F}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^r = \sqrt{G}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^\xi = \frac{1}{H}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \tag{8}$$

where σ's are Pauli matrices. For fermions, there are two states corresponding to spin up and spin down particles. The analysis for both cases is same, so in this paper, we consider only the spin up case. We assume the wave function as

$$\Psi(t, r, \xi) = \begin{pmatrix} A(t, r, \xi) \\ 0 \\ B(t, r, \xi) \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar}I(t, r, \xi)\right], \tag{9}$$

where I is action of emitted fermion. On substitution of the wave function (9) and the gamma matrices (8) into the modified Dirac equation (7), we get the action form of modified Dirac equation

$$-B(1-\beta m^2)\sqrt{G}\partial_r I + B\beta\sqrt{G}\partial_r I(g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)$$
$$- \frac{iA}{\sqrt{F}}\partial_t I - iA\frac{e\hat{A}_t}{\sqrt{F}}[1 - \beta m^2 - (g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)]$$
$$+ Am[(1-\beta m^2) - \beta(g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)] = 0, \tag{10}$$

$$
\begin{aligned}
-A(1-\beta m^2)\sqrt{G}\partial_r I + A\beta\sqrt{G}\partial_r I(g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2) \\
+\frac{iB}{\sqrt{F}}\partial_t I + iB\frac{e\hat{A}_t}{\sqrt{F}}[1-\beta m^2 - (g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)] \\
+Bm[(1-\beta m^2) - \beta(g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)] = 0, \quad (11)
\end{aligned}
$$

$$
A\sqrt{g^{\xi\xi}}\partial_\xi I[-(1-\beta m^2) + \beta(g^{rr}(\partial_r I)^2 + g^{\xi\xi}(\partial_\xi I)^2)] = 0. \quad (12)
$$

It is hard to directly solve the above system of coupled equations for action. Indeed, the line element (1), equivalently (5), is stationary and admit a Killing vector field ∂_t, so we decompose the action I as

$$
I = -(\omega - j\Omega)t + W(r,\xi), \quad (13)
$$

where ω and j are the energy and angular momentum of the emitted fermion. From (10) and (11), it is easy to see that for nonzero zero wave function, $\partial_\xi I = 0$, which just mean that in dragging coordinates action is independent of ξ. Thus, from now on without loss of generality we fix $\xi = \xi_0$. Inserting (13) into (10) and (11) we obtained

$$
\begin{aligned}
\frac{iA}{\sqrt{F}}(\omega - j\Omega) - iA\frac{e\hat{A}_t}{\sqrt{F}}[1-\beta m^2 - g^{rr}(\partial_r W)^2] + B\beta g^{rr}\sqrt{G}(\partial_r W)^3 \\
- B(1-\beta m^2)\sqrt{G}\partial_r W + Bm[(1-\beta m^2) - \beta g^{rr}(\partial_r W)^2] = 0, \quad (14)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{iB}{\sqrt{F}}(\omega - j\Omega) + iB\frac{e\hat{A}_t}{\sqrt{F}}[1-\beta m^2 - g^{rr}(\partial_r W)^2] + A\beta g^{rr}\sqrt{G}(\partial_r W)^3 \\
- A(1-\beta m^2)\sqrt{G}\partial_r W + Bm[(1-\beta m^2) - \beta g^{rr}(\partial_r W)^2] = 0. \quad (15)
\end{aligned}
$$

This is system of homogeneous equations in A and B and have nontrivial solution, if determinant of the coefficient matrix vanishes, thus we must have

$$
A_6(\partial_r W)^6 + A_4(\partial_r W)^4 + A_2(\partial_r W)^2 + A_0 = 0, \quad (16)
$$

where

$$
\begin{aligned}
A_6 &= \beta^2 G^3 F, \\
A_4 &= \beta G^2 F(m^2\beta - 2) - \beta^2 G^2 e^2 \hat{A}_t^2, \\
A_2 &= GF(1-\beta m^2)(1+\beta m^2) + 2\beta e\hat{A}_t G[-(\omega - j\Omega) + e\hat{A}_t(1-\beta m^2)], \\
A_0 &= -m^2 F(1-\beta m^2)^2 - [(\omega - j\Omega) - e\hat{A}_t(1-\beta m^2)]^2.
\end{aligned}
$$

Solving (16) by neglecting higher powers of β we get

$$
W_\pm = \pm\int\sqrt{\frac{m^2F + [(\omega - j\Omega) - e\hat{A}_t(1-\beta m^2)]^2}{FG}}\left[1 + \beta\frac{m^2F + \omega_0^2 - e\hat{A}_t\omega_0}{F}\right]dr, \quad (17)
$$

where $\omega_0 = \omega - j\Omega - e\hat{A}_t$ and $+/-$ corresponds to outgoing/ingoing solutions. The above integral equation has pole at horizons of the black hole and can be solved by complex contour integration. Note that (17) has pole of order 2 at horizons and thus instead of using $F(r) = (r - r_h)F'(r_h)$ and $G(r) = (r - r_h)G'(r_h)$ by Taylor theorem, we use factor theorem so that $F_1(r) = F(r)/(r - r_h)$ and $G_1(r) = G(r)/(r - r_h)$. Solving around the horizon r_h with fixed $\xi = \xi_0$, we get

$$\mathrm{Im}(W_\pm) = \pm\pi \frac{(\omega - j\Omega_h - eA_{t_h})}{\sqrt{F_1(r_h)G_1(r_h)}}(1 + \beta\Xi), \tag{18}$$

where

$$\Xi = \frac{3}{2}m^2 + \frac{m^2 e\hat{A}_{t_h}}{2\bar{\omega}_0} - \left(\frac{2e(2\bar{\omega}_0 - e\hat{A}_{t_h})\hat{A}'_{t_h} + j(3\bar{\omega}_0 - 2e\hat{A}_{t_h})\Omega'_h}{F_1(r_+)}\right) - \frac{1}{2}\frac{\bar{\omega}_0(\bar{\omega}_0 - e\hat{A}_{t_h})}{F_1(r_h)}\left(3\frac{F_1'(r_h)}{F_1(r_h)} + \frac{G_1'(r_h)}{G_1(r_h)}\right). \tag{19}$$

Here, $\hat{A}_{t_h} = \hat{A}_t(r_h)$, $\Omega_h = \Omega(r_h)$, $\bar{\omega}_0 = \omega - j\Omega_h - e\hat{A}_{t_h}$ and "prime" denotes derivative with respect to r. The tunneling probability of the fermions, with the contribution of temporal part is given as[85]

$$\Gamma \propto \exp[-(\mathrm{Im}(\bar{\omega}_0 \Delta t^{\mathrm{out,in}}) + \mathrm{Im}\, p_r\, dr)], \tag{20}$$

where

$$\mathrm{Im}(\bar{\omega}_0 \Delta t^{\mathrm{out,in}}) = \frac{\pi\bar{\omega}_0}{2\kappa_h},$$

with κ_h is the standard surface gravity of corresponding stationary axially symmetric black hole at horizon r_h. Considering total temporal contribution $\mathrm{Im}(\bar{\omega}_0\Delta t) = \pi\bar{\omega}_0/\kappa_h$, we get the expression of the tunneling probability

$$\Gamma = \exp[-(\mathrm{Im}(\bar{\omega}_0\Delta t) + 2\,\mathrm{Im}\, W^{\mathrm{out}})], \tag{21}$$

$$\Gamma = \exp\left[-4\pi\frac{(\omega - j\Omega_+ - eA_{t_+})}{\sqrt{F_1(r_+)G_1(r_+)}}\left(1 + \frac{\beta}{2}\Xi\right)\right]. \tag{22}$$

Thus, the modified Hawking temperature for black hole with line element (1) reads the value

$$T = \frac{1}{4\pi}\frac{\sqrt{F_1(r_h)G_1(r_h)}}{(1 + \frac{\beta}{2}\Xi)} = T_0\left(1 - \frac{\beta}{2}\Xi\right), \tag{23}$$

where $T_0 = \sqrt{F_1(r_h)G_1(r_h)}/4\pi$ is the standard Hawking temperature of (1). Note that for positive temperature $\frac{2}{\beta} > \Xi$ and for $\Xi > 0$ the modified temperature is lower then standard temperature.

3. Modified Temperature of Kerr–Newman Black Hole

In this section, we use the general formula derived for modified Hawking temperature in last section to find modified temperature of Kerr–Newman black hole. The Kerr–Newman black hole is stationary axially symmetric black hole and its line element share the form of (1) with $\xi = \theta$ as[88]

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\,dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}d\phi^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}dt\,d\phi, \tag{24}$$

with the electromagnetic potential

$$A_\mu = \frac{Qr}{\Sigma}dt - \frac{Qra\sin^2\theta}{\Sigma}d\phi, \tag{25}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 + Q^2 - 2Mr = (r - r_+)(r - r_-)$ and the parameter M, Q and a denote the mass, electric charge and angular momentum per unit mass, respectively. The outer and inner horizons are located at $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$. The angular velocity for the Kerr–Newman black hole given by

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a(r^2+a^2-\Delta)}{K}, \tag{26}$$

where $K = (r^2+a^2)^2 - \Delta a^2\sin^2\theta$. Using dragged coordinate transformation $d\phi = \Omega\,dt$, with angular velocity (26), the dragged line element and corresponding electromagnetic potential of Kerr–Newman black hole takes the form

$$d\hat{s}^2 = -\frac{\Sigma\Delta}{K}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2, \tag{27}$$

and

$$\hat{A}_i = \hat{A}_t\,dt = \frac{Qr(r^2+a^2)}{K}dt. \tag{28}$$

To determine modified temperature at outer horizon r_+ the functions F_1 and G_1 are

$$F_1 = \frac{(r-r_-)\Sigma}{K} \quad \text{and} \quad G_1 = \frac{(r-r_-)}{\Sigma}. \tag{29}$$

Now we are in position to find modified temperature for Kerr–Newman black hole using the formula (23) with correction terms given by (19). Using angular velocity (26), electromagnetic potential (28) and functions (29) into (19) we get

$$\Xi_{\mathrm{KN}} = \frac{3}{2}m^2 + \frac{m^2e\hat{A}_{t+}}{2\bar{\omega}_0} + \frac{2eQ(2\bar{\omega}_0 - e\hat{A}_{t+})}{\Sigma(r_+)(r_+-r_-)}\left[r_+^2 - a^2 - \frac{r_+(r_+-r_-)}{r_+^2+a^2}a^2\sin^2\theta_0\right]$$
$$+ \frac{(3\bar{\omega}_0 - 2e\hat{A}_{t+})}{\Sigma(r_+)(r_+-r_-)}[j\Omega_+\{4r_+(r_+^2+a^2) - (r_+-r_-)a^2\sin^2\theta_0\}$$

$$-ja(r_+ + r_-)] + \frac{\bar{\omega}_0(\bar{\omega}_0 - e\hat{A}_{t+})}{2\Sigma(r_+)(r_+ - r_-)}\left[12r_+(r_+^2 + a^2) - \frac{4(r_+^2 + a^2)^2}{r_+ - r_-}\right.$$

$$\left. - 3(r_+ - r_-)a^2 \sin^2\theta_0 - \frac{4r_+(r_+^2 + a^2)^2}{\Sigma(r_+)}\right], \tag{30}$$

with

$$\bar{\omega}_0 = \omega - j\Omega_+ - e\hat{A}_{t+}, \quad \Omega_+ = \frac{a}{r_+^2 + a^2}, \quad \hat{A}_{t+} = \frac{Qr_+}{r_+^2 + a^2}. \tag{31}$$

Thus, the modified Hawking temperature for Kerr–Newman black hole is

$$T = \frac{1}{4\pi}\frac{r_+ - r_-}{r_+^2 + a^2}\left(1 + \frac{1}{2}\beta\Xi_{\mathrm{KN}}\right)^{-1} = T_0\left(1 - \frac{1}{2}\beta\Xi_{\mathrm{KN}}\right), \tag{32}$$

where T_0 is standard Hawking temperature for Kerr–Newman black hole. When $a = 0$ and $q = 0$, the modified temperature reduces to the Reissner–Nordström and Kerr black holes, respectively. Due to θ_0 in the correction term, the modified temperature depends on angle θ. Their claim is not in agreement with zeroth law of thermodynamics. So, for constant temperature everywhere on the horizons we can set $\theta_0 = 0$, and in this case correction terms Ξ_{KN} reduces to

$$\Xi_{\mathrm{KN}} = \frac{3}{2}m^2 + \frac{m^2 e\hat{A}_{t+}}{2\bar{\omega}_0} + \frac{2eQ(2\bar{\omega}_0 - e\hat{A}_{t+})(r_+^2 - a^2)}{(r_+ - r_-)(r_+^2 + a^2)}$$

$$+ \frac{aj(3\bar{\omega}_0 - 2e\hat{A}_{t+})}{(r_+ - r_-)(r_+^2 + a^2)}[4r_+(r_+^2 + a^2) - (r_+ + r_-)]$$

$$+ \frac{2\bar{\omega}_0(\bar{\omega}_0 - e\hat{A}_{t+})}{(r_+ - r_-)}\left[2r_+ - \frac{(r_+^2 + a^2)}{r_+ - r_-}\right]. \tag{33}$$

Using $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$ it can be easily shown that $\Xi_{\mathrm{KN}} > 0$, thus the modified temperature is lower than that of standard temperature and for positive temperature it must be in the limit $\Xi_{\mathrm{KN}} < \frac{2}{\beta}$. Further, if we ignore the quantum gravity effects we will get the standard temperature for Kerr–Newman black holes.

4. Modified Temperature for EMDA Black Hole

In this section we give the modified temperature for EMDA black hole. The EMDA black hole is stationary axially symmetric solution of the EMDA field equations. The line element of EMDA black hole of mass M, angular momentum per unit mass a and dilatonic perimeter b is given by[1]

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 - 2\frac{a\sin^2\theta(r^2 + 2br + a^2 - \Delta)}{\Sigma}dt\, d\phi$$

$$+ \frac{(r^2 + 2br + a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\, d\phi^2. \tag{34}$$

The corresponding electromagnetic vector potential is

$$A_\mu = \frac{Qr}{\Sigma}dt - \frac{Qra\sin^2\theta}{\Sigma}d\phi, \tag{35}$$

where $\Sigma = r^2 + 2br + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$ and $r_\pm = M \pm \sqrt{M^2 - a^2}$ are location of the outer and inner horizons. The ADM mass M_A, charge Q and the angular momentum J are related with diatonic parameter as

$$M_A = M + b, \quad Q^2 = 2b(M + b), \quad J = (M + b)a.$$

The EMDA black hole is generalization of the Kerr and the Garfinkle–Horowitz–Strominger dilatonic (GHSD) black holes, with parameter b and a, respectively. The angular velocity for this black hole is given as

$$\Omega = \frac{a(r^2 + 2br + a^2 - \Delta)}{K}, \tag{36}$$

with $K = (r^2 + 2br + a^2)^2 - \Delta a^2\sin^2\theta$. With this angular velocity we obtained the transformed dragged line element as

$$d\hat{s}^2 = -\frac{\Delta\Sigma}{K}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2, \tag{37}$$

and the electromagnetic potential

$$A_i = \hat{A}_t\, dt = \frac{Qr(r^2 + 2br + a^2)}{K}dt. \tag{38}$$

Using the factor theorem for outer horizon, we can define the functions as

$$F_1 = \frac{(r - r_-)\Sigma}{K} \quad \text{and} \quad G_1 = \frac{(r - r_-)}{\Sigma}. \tag{39}$$

Using angular velocity (36), electromagnetic potential (38) and functions (39) we have

$$\begin{aligned}\Xi_{\text{EDMA}} = {}& \frac{3}{2}m^2 + \frac{m^2 e\hat{A}_{t+}}{2\bar{\omega}_0} + \frac{2eQ(2\bar{\omega}_0 - e\hat{A}_{t+})(r_+^2 - a^2)}{\Sigma(r_+)(r_+ - r_-)} \\ & + \frac{aj(3\bar{\omega}_0 - 2e\hat{A}_{t+})}{(r_+ - r_-)}\frac{[4(r_+ + b) - (r_+ + 2b + r_-)]}{(r_+^2 + 2br_+ + a^2)} \\ & + \frac{\bar{\omega}_0(\bar{\omega}_0 - e\hat{A}_{t+})}{(r_+ - r_-)}\left[4(r_+ + b) - \frac{2(r_+^2 + 2br_+ + a^2)}{r_+ - r_-}\right],\end{aligned} \tag{40}$$

with

$$\begin{aligned}\bar{\omega}_0 = \omega - j\Omega_+ - e\hat{A}_{t+}, \quad \Omega_+ = \tfrac{a}{r_+^2 + 2br_+ + a^2}, \\ A_{t+} = \frac{Qr_+}{r_+^2 + 2br_+ + a^2}.\end{aligned} \tag{41}$$

Thus modified Hawking temperature for EDMA black hole is given as

$$T = \frac{1}{4\pi}\left(\frac{r_+^2 + 2br_+ + a^2}{r_+ - r_-}\right)\frac{1}{\left(1 + \frac{1}{2}\Xi_{\text{EDMA}}\right)} = T_0\left(1 - \frac{1}{2}\beta\Xi_{\text{EDMA}}\right). \tag{42}$$

When $a = 0$, we will get modified temperature for GHSD black holes. Further, if $\Xi_{\text{EDMA}} > 0$, the modified temperature is less than the standard temperature.

5. Modified Temperature for Kaluza–Klein Dilaton Black Hole

The Kaluza–Klein black hole is an exact solution of the dilatonic action with coupling constant $\alpha = 3$. It is derived by a dimensional reduction of the boosted five dimensional Kerr solution to four dimensions. The line element of Kaluza–Klein dilaton black hole is given by[2,3]

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2\sin^2\theta}{\Pi\Sigma}dt^2 + \frac{\Pi\Sigma}{\Delta}dr^2 + \Pi\Sigma\, d\theta^2 - 2\frac{aZ\sin^2\theta}{\Pi\sqrt{1-\nu^2}}dt\, d\phi \\ &\quad + \left[\Pi(r^2 + a^2) + \frac{Z}{\Pi}a^2\sin^2\theta\right]\sin^2\theta\, d\phi^2, \end{aligned} \tag{43}$$

with the electromagnetic potential and the dilaton field

$$A_\mu = \frac{\nu}{2(1-\nu^2)}\frac{Z}{\Pi^2}dt - \frac{\nu a\sin^2\theta}{2\sqrt{1-\nu^2}}\frac{Z}{\Pi^2}d\phi, \tag{44}$$

$$\Phi = -\frac{\sqrt{3}\ln\Pi}{2}, \tag{45}$$

where

$$Z = \frac{2\mu r}{\Sigma}, \quad \Pi = \sqrt{1 + \frac{\nu^2 Z}{1-\nu^2}}, \tag{46}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \tag{47}$$

and M, a and ν are the mass parameter specific angular momentum and boosted velocity, respectively. The horizons are located at $r_\pm = M \pm \sqrt{\mu^2 - a^2}$. The physical mass M_p, the charge Q and the angular momentum J can be related with mass parameter and boosted velocity and specific angular momentum as

$$M_p = M\left[1 + \frac{\nu^2}{2(1-\nu^2)}\right], \quad Q = \frac{M\nu}{1-\nu^2}, \quad J = \frac{Ma}{\sqrt{1-\nu^2}}. \tag{48}$$

The angular velocity for this black hole

$$\Omega = \frac{aZ\sqrt{1-\nu^2}}{(r^2 + a^2)Z + (1-\nu^2)\Delta}. \tag{49}$$

Using this angular velocity the dragged line element becomes

$$d\hat{s}^2 = -\frac{\Pi\Sigma\Delta(1-v^2)}{K}dt^2 + \frac{\Pi\Sigma}{\Delta}dr^2 + \Pi\Sigma\, d\theta^2, \tag{50}$$

with $K = (r^2+a^2)^2 - \Delta(a^2\sin^2\theta + \nu^2\Sigma)$. The dragged electromagnetic potential is

$$\hat{A}_i = \hat{A}_t\,dt = \frac{Qr(r^2+2br+a^2)}{K}dt. \tag{51}$$

For modified temperature at horizon r_+ the functions are

$$F_1 = \frac{\Pi\Sigma(1-\nu^2)(r-r_-)}{K} \quad \text{and} \quad G_1 = \frac{(r-r_-)}{\Pi\Sigma}. \tag{52}$$

Using angular velocity (49), electromagnetic potential (51) and the functions (52) we get

$$\begin{aligned}\Xi_{\rm KK} = {}& \frac{3}{2}m^2 + \frac{m^2 e\hat{A}_{t_+}}{2\omega_0} - 2\frac{e\hat{A}_{t_+}(2\omega_0 - e\hat{A}_{t_+})}{F_1(r_\alpha)}\left[\frac{2r}{r^2+a^2} + \frac{1}{r} - \frac{2r}{\Sigma}\right.\\ &\left. - \frac{2\Pi'_+(r_+^2+a^2)+4r_+\Pi_+}{\Pi_+(r_+^2+a^2)}\right] - \frac{1}{2}\frac{\omega_0(\omega_0 - e\hat{A}_{t_+})}{\Pi_+(1-\nu^2)(r_+-r_-)(r_+^2+a^2)}\\ &\times\left[\frac{4(r_+^2+a^2)^2}{r_+-r_-} + \frac{2m\nu^2(a^2-r_+^2)}{[1+\nu^2(Z_+-1)]} - (r_+^2+a^2)\{8r_+ + 6(r_+-\mu)\nu^2\}\right]\\ &+ \frac{aj(3\omega_0 - 2e\hat{A}_{t_+})}{\sqrt{1-\nu^2}}\left[\frac{2r_+Z_+ + (1-\nu^2)(r_+-r_-)}{Z_+\Pi_+(r_+-r_-)(r_+^2+a^2)}\right],\end{aligned} \tag{53}$$

where

$$\bar{\omega}_0 = \omega - j\Omega_+ - e\hat{A}_{t+}, \quad \Omega_+ = \frac{a\sqrt{1-\nu^2}}{r^2+a^2}, \quad A_{t+} = \frac{Qr_+}{r_+^2+2br_++a^2}. \tag{54}$$

6. Modified Temperature for Charged Rotating Black Strings

The line element of charged rotating black string is stationary and axially symmetric which admit three Killing vectors, ∂_t, ∂_ϕ, ∂_z. Thus the modified temperature for black string can be obtained from general formula given in Sec. 2 with $\xi = z$. The line element of charged rotating black strings can be written as[90]

$$ds^2 = -\Delta\left(\gamma\,dt - \frac{\delta}{\alpha^2}d\phi\right)^2 + r^2(\gamma\,d\phi - \delta\,dt)^2 + \frac{dr^2}{\Delta} + \alpha^2 r^2\,dz^2, \tag{55}$$

with

$$\Delta = \alpha^2r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2r^2}, \quad b = 4M\left(1 - \frac{3a^2\alpha^2}{2}\right), \quad c^2 = 4Q^2\left(\frac{2-3a^2\alpha^2}{2-a^2\alpha^2}\right), \tag{56}$$

$$\gamma = \sqrt{\frac{2-a^2\alpha^2}{2-2a^2\alpha^2}}, \quad \delta = \frac{a\alpha^2}{\sqrt{1-\frac{3}{2}a^2\alpha^2}}. \tag{57}$$

where M is mass, a is the angular momentum per unit mass, $\alpha^2 = -\frac{\Lambda}{3}$, Λ is the negative cosmological constant. The parameters a and α^2 are related with angular momentum J and the mass of the black hole M as

$$a^2\alpha^2 = 1 - \frac{\sqrt{M^2 - \frac{8J^2\alpha^2}{9}}}{M}, \tag{58}$$

$$J = \frac{3}{2}Ma\sqrt{1 - \frac{a^2\alpha^2}{2}}. \tag{59}$$

The corresponding electromagnetic potential is given by

$$A_\mu = \frac{2Q}{\alpha r}dt - \frac{2\delta Q}{\alpha^3 r\gamma}d\phi. \tag{60}$$

The angular velocity for charged rotating black string is

$$\Omega = \frac{-\Delta\gamma\delta\alpha^2 + r^2\gamma\delta\alpha^4}{-\Delta\delta^2 + r^2\gamma^2\alpha^4}. \tag{61}$$

With this angular velocity after transformation, we get dragged metric[91]

$$\begin{aligned} d\hat{s}^2 &= -\frac{\Delta r^2\alpha^4}{-\Delta\delta^2 + r^2\gamma^2\alpha^4}dt^2 + \frac{dr^2}{\Delta} + \alpha^2 r^2\,dz^2, \\ d\hat{s}^2 &\equiv -F\,dt^2 + \frac{dr^2}{G} + H^2\,d\xi^2. \end{aligned} \tag{62}$$

The corresponding electromagnetic vector potential is

$$\hat{A}_i = \hat{A}_t\,dt = \left(\frac{2Qr\alpha^3}{-\Delta\delta^2 + r^2\gamma^2\alpha^4}\right)dt. \tag{63}$$

For modified temperature at horizon of black string r_+, we have

$$\begin{aligned} F_1 &= \frac{\alpha^2\left(\alpha^4 r^3 + \alpha^4 r_+ r^2 + \alpha^4 r_+^2 r - \frac{c^2}{r_+}\right)}{-\Delta\delta^2 + r^2\gamma^2\alpha^4}, \\ G_1 &= \alpha^2 r + \alpha^2 r_+ + \frac{\alpha^2 r_+^2}{r} - \frac{c^2}{\alpha^2 r_+ r^2}. \end{aligned} \tag{64}$$

With angular velocity (61), (63) and (64)

$$\begin{aligned} \Xi_{\mathrm{BS}} &= \frac{3}{2}m^2 + \frac{m^2 e\hat{A}_{t_+}}{2\bar{\omega}_0} + \frac{4eQ(2\bar{\omega}_0 - e\hat{A}_{t_+})}{\alpha r_+^2\Delta'(r_+)}\left(1 - \frac{\Delta'(r_+)\delta^2}{r_+\gamma^2\alpha^4}\right) \\ &\quad + \frac{j(3\bar{\omega}_0 - 2e\hat{A}_{t_+})\Omega_+}{\alpha^2 r_+^2} - \frac{1}{2}\frac{\bar{\omega}_0(\bar{\omega}_0 - e\hat{A}_{t_+})\gamma^2}{\Delta'(r_+)} \\ &\quad \times\left[3\left(\frac{6\alpha^2}{\Delta'(r_+)} + \frac{\delta^2}{r_+^2\gamma^2\alpha^4}\Delta'(r_+) - \frac{2}{r_+}\right) - \frac{\alpha^2 r_+^4}{2c^2}\Delta'(r_+)\right], \end{aligned} \tag{65}$$

where

$$\begin{aligned} \bar{\omega}_0 = \omega - j\Omega_+ - e\hat{A}_{t+}, \quad \Omega_+ = \frac{\delta}{\gamma}, \quad \hat{A}_{t_+} = \frac{2Q}{\alpha\gamma^2 r_+}, \\ \Delta'(r_+) = \frac{3\alpha^4 r_+^4 - c^2}{\alpha^2 r_+^3}. \end{aligned} \tag{66}$$

For $a = 0$, the modified temperatures for charged non-rotating black string and $Q = 0$ for uncharged rotating black string are successfully recovered.[84]

7. Conclusion

In this paper, we first developed a general method to study the tunneling of charged fermions from the stationary axially symmetric black holes with GUP. The important results are given as follows.

- To this end, we modified the Dirac equation using the GUP and solve it using the corresponding curved spacetime via the semiclassical method of Wentzel–Kramers–Brillouin (WKB) and Hamilton–Jacobi approach.
- After we obtained the corrected tunneling rate of the fermions from the curved spacetime, we showed the modified Hawking temperature for the most general case, then using this method, we gave same examples how to calculate Hawking temperatures of Kerr–Newman black hole, EMDA black hole, Kaluza–Klein dilaton black hole, and then, charged rotating black string.
- The corrected thermal spectrum was shown that it is not purely thermal. We noted that the effect of the GUP causes the black hole's remnant.
- Moreover, the modified Hawking temperature of the black holes is lower than the standard Hawking temperature.
- The remnant of the black hole's radiation increases, when the black hole size is close to the Planck scale, because of the effect of the quantum gravity.
- Due to this remnant, the black hole is prevented from evaporation, and its information and singularity are enclosed in the event horizon.

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