Exact traversable wormhole solution in bumblebee gravity

Ali Övgün,[1,2,*] Kimet Jusufi,[3,4,†] and İzzet Sakallı[2,‡]

[1]*Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile*
[2]*Physics Department, Arts and Sciences Faculty, Eastern Mediterranean University, Famagusta 99628, North Cyprus via Mersin 10, Turkey*
[3]*Physics Department, State University of Tetovo, Ilinden Street nn, 1200, Tetovo, Macedonia*
[4]*Institute of Physics, Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University, Arhimedova 3, 1000 Skopje, Macedonia*

(Received 28 April 2018; revised manuscript received 12 December 2018; published 29 January 2019)

In this study, we found a new traversable wormhole solution in the framework of a bumblebee gravity model. With these types of models, the Lorentz symmetry violation arises from the dynamics of a bumblebee vector field that is nonminimally coupled with gravity. To this end, we checked the wormhole's flare-out and energy (null, weak, and strong) conditions. We then studied the deflection angle of light in the weak limit approximation using the Gibbons-Werner method. In particular, we show that the bumblebee gravity effect leads to a nontrivial global topology of the wormhole spacetime. By using the Gauss-Bonnet theorem (GBT), it is shown that the obtained non-asymptotically flat wormhole solution yields a topological term in the deflection angle of light. This term is proportional to the coupling constant, but independent from the impact factor parameter. Significantly, we showed that the bumblebee wormhole solutions, under specific conditions, support the normal matter wormhole geometries.

DOI: 10.1103/PhysRevD.99.024042

I. INTRODUCTION

The search for a theory of wormholes through Einstein's general theory of relativity goes back to 1916 with the famous papers of Flamm [1]. The simplest possible solution to Einstein's field equations is the Schwarzschild metric [2], which describes the gravitational field around a spherically symmetric static mass. If the mass (or its density) is sufficiently high, the solution describes a black hole—the Schwarzschild black hole. Flamm realized that Einstein's equations also allow a second solution, which is presently known as a "white hole." These two solutions, describing two different regions of (flat) spacetime, are connected by a "spacetime tube." This tube does not define where those regions of spacetime might be in the real world; the black hole's "entrance" and white hole's "exit" could exist in different portions of the same universe or in entirely different universes. In 1935, Einstein and Rosen [3] further explored the theory of interuniverse connections. In fact, their main aim was to try to understand the fundamental charged particles (protons, electrons, etc.) in terms of spacetime tubes penetrated by lines of electromagnetic force. These spacetime passageways were named "Einstein-Rosen Bridges" by Wheeler, who would later call them *wormholes*. It is worth noting that Wheeler [4] also coined the term "black hole." Traversable wormholes have no horizon and allow two-way traveling [5] by connecting two different regions of spacetime in a Lorentzian geometry. Interest in traversable wormhole gained momentum following the paper of Morris, Thorne, and Yurtsever (MTY) [6] as shown in Fig. 1. With a traversable wormhole, an interstellar or interuniverse journey is possible [7,8]. However, to construct such a traversable wormhole, one requires an exotic matter with a negative energy density and a large negative pressure, which should have a higher value than the energy density. Meanwhile, the Casimir effect [9] is a way of producing negative energy density. MTY also proved that traversable wormholes could be stabilized using the Casimir effect. Toward this end, placing two sufficiently charged superconducting spheres at the traversable wormhole mouths is enough. On the other hand, in 2011, Kanti and Kleihaus [10] showed that it might be possible to construct a traversable wormhole using normal matter by resorting to a form of string theory.

In the literature, many authors have intensively studied various aspects of traversable wormhole geometries within different modified gravitational theories [11–60]. Among them, the bumblebee gravity model has dynamically violated Lorentz symmetry in terms of charge conjugation, parity transformation, and time reversal. The model, with its defined bumblebee vector field, can also feature rotation and boost [61–66]. In fact, bumblebee gravity was first used by Kostelecky and Samuel in 1989 [67,68] as a simple

*ali.ovgun@pucv.cl; https://aovgun.weebly.com
†kimet.jusufi@unite.edu.mk
‡izzet.sakalli@emu.edu.tr

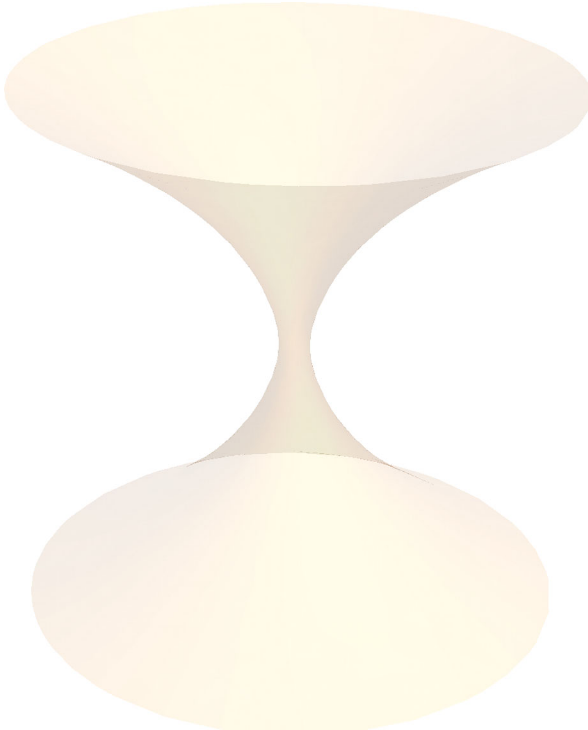

FIG. 1. Traversable wormhole.

model for investigating the consequences of spontaneous Lorentz violation.

The bumblebee mechanism arose in the context of string theory and lead to a spontaneous breaking of Lorentz symmetry by tensor-valued fields acquiring vacuum expectation values [65]. The forcefulness of the bumblebee vector field on the gravitational field has motivated us to construct traversable wormholes. Very recently, a Schwarzschild-like bumblebee black hole solution has been obtained [69]. From the perspective of string theory and loop quantum gravity theory, Lorentz symmetry breaking (LSB) is an interesting idea for exploring the tracks of the quantum gravity at low energy levels. LSB has been extensively studied in the literature, e.g., see [70–75] and references therein.

The main aim of this paper is to construct an exact solution of a traversable wormhole in the bumblebee gravity field, where in Einstein's field equations are influenced by spontaneous breaking of Lorentz symmetry. We sought to compute the weak deflection angle of the obtained bumblebee wormhole. To this end, we employed the Gibbons-Werner method (GWM) [76]. In this method, the deflection angle, for the weak lensing limit, is calculated by the Gauss-Bonnet theorem (GBT), defined by the background optical geometry. It is important to highlight that non-singular domains are considered to be outside of the light ray, which means that the GBT has a global impact [76–79]. Wormholes have been widely studied by many authors as have black holes; light deflection has been of particular interest [80–99]. Another purpose of this paper is to discover a traversable wormhole using normal matter, which satisfies energy and flare-out conditions in the bumblebee gravity. In the following sections, we shall explain how these goals are achieved.

This paper is organized as follows: In Sec. II, we briefly outline bumblebee gravity and its corresponding Einstein's field equations. In Sec. III, we present LSB wormhole solutions and study the flare-out conditions. We check the energy conditions of the bumblebee wormhole in Sec. IV. In the framework of the GWM, Sec. V is devoted to the study of the deflection angle of light in the weak limit approximation. Our conclusions and remarks follow in Sec. VI.

II. BUMBLEBEE GRAVITY

The action of the bumblebee gravity where the Lorentz violation arises from the dynamics of a single vector field, namely bumblebee field B_μ, with a real coupling constant ξ (with mass dimension −1) is given by

$$S_B = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R + \frac{1}{2\kappa}\xi B^\mu B^\nu R_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - V(B^\mu)\right] + \int d^4x \mathcal{L}_{\mathrm{M}}, \tag{2.1}$$

where the bumblebee field strength ($B_{\mu\nu}$) and the potential (V) are defined as follows

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\mu B_\nu, \tag{2.2}$$

$$V \equiv V(B^\mu B_\mu \pm a^2). \tag{2.3}$$

where a^2 is a positive real constant [69]. Vacuum expectation value of the bumblebee gravitational field is governed by the following condition

$$V(B^\mu B_\mu \pm a^2) = 0.$$

This automatically implies that

$$B^\mu B_\mu \pm a^2 = 0, \tag{2.4}$$

in which the signs (±) potentially determines the field type of b_μ: timelike or spacelike. Solutions of Eq. (2.4) are conditional on the field B^μ that acquires a non-null vacuum expectation value:

$$\langle B^\mu \rangle = b^\mu. \tag{2.5}$$

In this setup, we use null torsion and null cosmological constant, so that there is a non-null vector b^μ which satisfies $b^\mu b_\mu = \mp a^2 = \alpha = \text{constant}$. Thus, the nonzero vector background b^μ, which is a coefficient for Lorentz and

CPT violation, spontaneously breaks the $U(1)$ symmetry [100].

Bumblebee modified Einstein field equations [69] are governed by

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.6}$$

where the total energy-momentum tensor is given by [100]

$$T_{\mu\nu} = T^M_{\mu\nu} + T^B_{\mu\nu}, \tag{2.7}$$

in which $T^M_{\mu\nu}$ is the matter field and the bumblebee energy-momentum tensor $T^B_{\mu\nu}$ reads

$$\begin{aligned} T^B_{\mu\nu} = & -B_{\mu\alpha}B^\alpha_\nu - \frac{1}{4}B_{\alpha\beta}B^{\alpha\beta}g_{\mu\nu} - Vg_{\mu\nu} + 2V'B_\mu B_\nu \\ & + \frac{\xi}{\kappa}\left[\frac{1}{2}B^\alpha B^\beta R_{\alpha\beta}g_{\mu\nu} - B_\mu B^\alpha R_{\alpha\nu} - B_\nu B^\alpha R_{\alpha\mu} \right. \\ & \left. + \frac{1}{2}\nabla_\alpha\nabla_\mu(B^\alpha B_\nu) - \frac{1}{2}\nabla^2(B_\mu B_\nu) - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(B^\alpha B^\beta)\right]. \end{aligned} \tag{2.8}$$

Thus, the modified Einstein field equations (2.6) with the bumblebee field can be expressed as follows [69]

$$R_{\mu\nu} - 8\pi G\left[T^M_{\mu\nu} + T^B_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(T^M + T^B)\right] = 0, \tag{2.9}$$

which has the following explicit form:

$$\begin{aligned} E^{\text{instein}}_{\mu\nu} = & R_{\mu\nu} - \kappa\left(T^M_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^M\right) - \kappa T^B_{\mu\nu} - 2\kappa g_{\mu\nu}V \\ & + \kappa B_\alpha B^\alpha g_{\mu\nu}V' - \frac{\xi}{4}g_{\mu\nu}\nabla^2(B_\alpha B^\alpha) \\ & - \frac{\xi}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(B^\alpha B^\beta) = 0, \end{aligned} \tag{2.10}$$

where $T^M = g^{\mu\nu}T^M_{\mu\nu}$ and $\kappa = 8\pi$. The prime denotes a derivative with respect to r. It can be easily checked that when the both bumblebee field B_μ and potential $V(B_\mu)$ are vanished, the original general relativity field equations are recovered.

Here, we focus on the vacuum solutions induced by the LSB, which is possible when the bumblebee field B_μ remains frozen in its vacuum expectation value. Namely, we consider the case of Eq. (2.5), so that we have a vanishing potential: $V = V' = 0$ [101]. Thus, the bumblebee field strength (2.2) becomes

$$b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu. \tag{2.11}$$

III. EXACT SOLUTION OF BUMBLEBEE WORMHOLE

In this section, we consider a static and a spherically symmetric traversable wormhole solution [5] without any tidal force

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{W(r)}{r}} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{3.1}$$

where $W(r)$ is the shape function of the wormhole. Furthermore, we set the bumblebee vector as follows

$$b_\mu = \left[0, \sqrt{\frac{\alpha}{1 - \frac{W(r)}{r}}}, 0, 0\right]. \tag{3.2}$$

The bumblebee modified Einstein's field equations with the isotropic matter $(T^\mu{}_\nu)^M = (-\rho, p, p, p)$ [51]. We shall use the equation of state: $p = w\rho$, in which ρ denotes the energy density of the matter field, p stands for the pressure, and w is the dimensionless $\mathbb{R}$ number. Thus, Eq. (2.10) yields the following three Einstein's field equations in the bumblebee gravity theory for the wormhole metric (3.1) (the details are tabulated in the Appendix):

$$E^{\text{instein}}_{tt} = -\kappa\rho r^3 - 3\kappa w\rho r^3 + \xi\alpha rW' - \xi\alpha W(r) = 0, \tag{3.3}$$

$$\begin{aligned} E^{\text{instein}}_{rr} = & 2rW' - 2W(r) + \kappa w\rho r^3 - \kappa\rho r^3 \\ & + 3\xi\alpha rW' - 3\xi\alpha W(r) = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} E^{\text{instein}}_{\theta\theta} = & \kappa w\rho r^3 - \kappa\rho r^3 + 2\xi\alpha W(r) - 2\xi\alpha r \\ & + rW' + W(r) = 0. \end{aligned} \tag{3.5}$$

From Eq. (3.3), one can obtain the density as follows (see Appendix)

$$\rho = \frac{l[rW' - W(r)]}{\kappa r^3(1 + 3w)}, \tag{3.6}$$

in which $l = \xi\alpha$. Solving Eq. (3.5) with Eq. (3.6), we find the shape function as follows

$$W(r) = \frac{lr}{l+1} + \frac{r_0}{l+1}\left(\frac{r_0}{r}\right)^{\frac{5wl+3l+3w+1}{wl-l+3w+1}}, \tag{3.7}$$

where $W(r_0) = r_0 \neq 0$: throat radius. Inserting Eq. (3.7) into Eq. (3.4), we get a condition on ω as the following

$$w = -\frac{l+1}{5l+3}. \tag{3.8}$$

Thus, the shape function and density become

$$W(r) = \frac{1}{l+1}\left(lr + r_0\left(\frac{r_0}{r}\right)^{\frac{-5l-3}{3l+1}}\right), \tag{3.9}$$

$$\rho = \frac{5l+3}{\kappa(3l+1){r_0}^2}(r^2 r_0)^{\frac{-2l}{3l+1}}. \tag{3.10}$$

It can be easily checked that as $l \to 0 \Rightarrow w \to -\frac{1}{3}$, $W(r) \to \frac{r^3}{{r_0}^2}$, and $\rho \to \frac{3}{\kappa {r_0}^2}$. On the other hand, one can easily see that g_{rr} diverges at $r = r_0$. Furthermore, Eq. (3.9) is nonasymptotically flat when ($r \to \infty$):

$$\lim_{r\to\infty}\frac{W(r)}{r} \to \frac{l}{l+1} + \lim_{r\to\infty}\left(\frac{r_0}{r}\right)^{1+\frac{-5l-3}{3l+1}}. \tag{3.11}$$

From the above equation, we infer that the first term is independent of r, while the second term is vanished if $1 + \frac{-5l-3}{3l+1} > 0$. Non-asymptotically flatness reflects the non-trivial topological structure (arising from the LSB effects) of the wormhole. Such solutions are similar to the spacetimes whose having topological defects and dilaton fields [102–104]. However, in principle, the solution obtained should be matched to an exterior vacuum solution (for details, a reader may refer to [51]).

The Ricci scalar results in

$$R = \frac{3W'r^2 - 2W(r)W' + 3W(r)^2}{2r^6}. \tag{3.12}$$

At $r = r_0$, Eq. (3.12) results in:

$$R|_{r_0} = \frac{18l^2 + 24l + 12}{{r_0}^4(3l+1)^2}. \tag{3.13}$$

In a similar way, the Kretschmann scalar:

$$K = 2\frac{r^2W'^2 - 2rW(r)W' + 3W(r)^2}{r^6}, \tag{3.14}$$

yields

$$K|_{r_0} = \frac{36l^2 + 24l + 12}{{r_0}^4(3l+1)^2}. \tag{3.15}$$

It is clear from Eqs. (3.13) and (3.15) that singularity arises if $l = -\frac{1}{3}$.

A. Flare-out conditions:

Traversability of a wormhole is determined by the flare-out conditions [5]. We can easily visualize the spatial geometry of the wormhole using an embedding diagram. The metric with $t = t_0$ (constant) reduces to the following form at the equatorial plane $\theta = \frac{\pi}{2}$: [105]

$$ds^2 = \frac{dr^2}{1 - \frac{W(r)}{r}} + r^2 d\varphi^2. \tag{3.16}$$

Then, we embed the wormhole geometry into a Euclidean 3-space:

$$d\sigma^2 = dz^2 + dr^2 + r^2 d\varphi^2, \tag{3.17}$$

which can be rewritten as follows

$$d\sigma^2 = (1 + z'^2)dr^2 + r^2 d\varphi^2, \tag{3.18}$$

where

$$z' = \pm\frac{1}{\sqrt{\frac{r}{W(r)} - 1}}. \tag{3.19}$$

We can now calculate the proper radial distance, which ought to be real and finite:

$$d(r) = \int_{r_0}^{r}\frac{dr}{\sqrt{1 - \frac{W(r)}{r}}}. \tag{3.20}$$

We deduce from the above equation that

$$\sqrt{1 - \frac{W(r)}{r}} > 0. \tag{3.21}$$

It is worth noting that there is a coordinate singularity at the throat of the wormhole. Thus, the flare-out conditions [5,17] yield:

$$W(r) - r \le 0, \tag{3.22}$$

and

$$rW' - W(r) < 0. \tag{3.23}$$

Thus, we have

$$W' = \frac{3l+3}{3l+1} < 1. \tag{3.24}$$

For the condition (3.24), it is depicted in Fig. 2 that the flare-out conditions are satisfied for the negative l values.

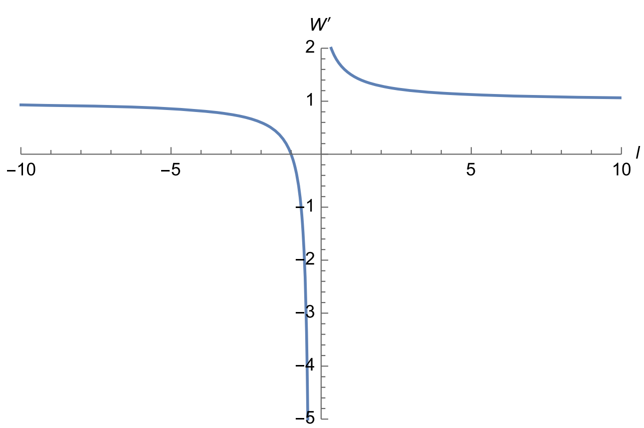

FIG. 2. W' versus l graph. The flare-out conditions are satisfied for the negative values of l.

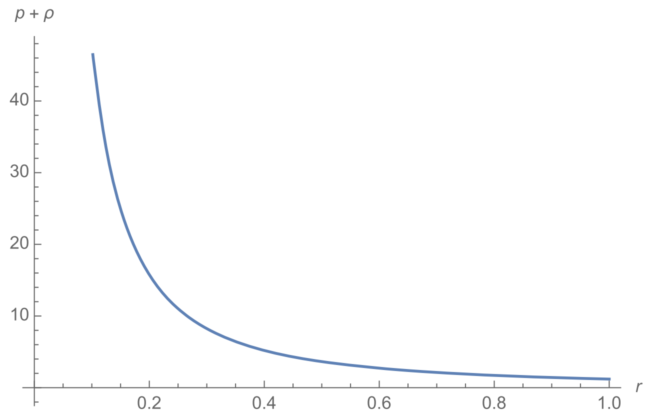

FIG. 3. Null energy condition $\rho + P \geq 0$ is satisfied for $r_0 = 1$, $l = -2$ and $\kappa = 1$.

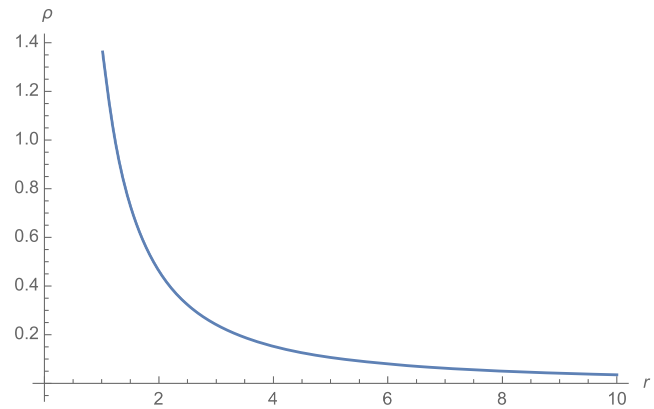

FIG. 4. The plot of energy density ρ for the values of parameters $r_0 = 1$, $l = -2$, and $\kappa = 1$.

IV. ENERGY CONDITIONS

Following the monograph of [106], in this section, we shall analyze the energy conditions for the bumblebee wormhole described with Eqs. (3.1) and (3.9).

A. Null energy condition:

The null energy condition is expressed in terms of energy density and pressure as follows

$$\rho + p \geq 0, \tag{4.1}$$

which yields

$$\rho + p = \frac{(4l+2)r_0}{(3l+1)\kappa r^3}\left(\frac{r_0}{r}\right)^{\frac{-5l-3}{3l+1}} \geq 0. \tag{4.2}$$

At $l = -\frac{1}{2}$, the null energy condition (4.2) becomes zero. It is clear from Fig. 3 that the null energy condition for the bumblebee wormhole is satisfied. Moreover, we depict an interactive plot in [107] for the null energy condition of the bumblebee wormhole in order to present the effect of parameter l.

B. Weak energy condition

Weak energy condition is given by

$$\rho \geq 0, \qquad \rho + p \geq 0, \tag{4.3}$$

which gives the following result for the bumblebee wormhole:

$$\rho = \frac{r_0(5l+3)}{(3l+1)\kappa r^3}\left(\frac{r_0}{r}\right)^{\frac{-5l-3}{3l+1}} \geq 0. \tag{4.4}$$

In Fig. 4, we show that weak energy condition for the bumblebee wormhole is satisfied when the physical parameters are fixed to $r_0 = 1$, $l = -2$, and $\kappa = 1$. One can reach to the interactive plot of the weak energy condition for the bumblebee wormhole with the link given in [107]. By this way, the effect of l on the weak energy condition can be monitored.

C. Strong energy condition

Strong energy condition is governed by

$$\rho + 3p \geq 0, \qquad \rho + p \geq 0, \tag{4.5}$$

which yields the following expression for the wormhole of bumblebee gravity:

$$\rho + 3p = \frac{2r_0 l}{(3l+1)\kappa r^3}\left(\frac{r_0}{r}\right)^{\frac{-5l-3}{3l+1}} \geq 0 \tag{4.6}$$

From Fig. 5, it can be seen that strong energy condition for the bumblebee wormhole is satisfied for the parameters of $r_0 = 1$, $l = -2$, and $\kappa = 1$. To reach to the interactive plot of the strong energy condition for the bumblebee wormhole, one can follow the link given in [108].

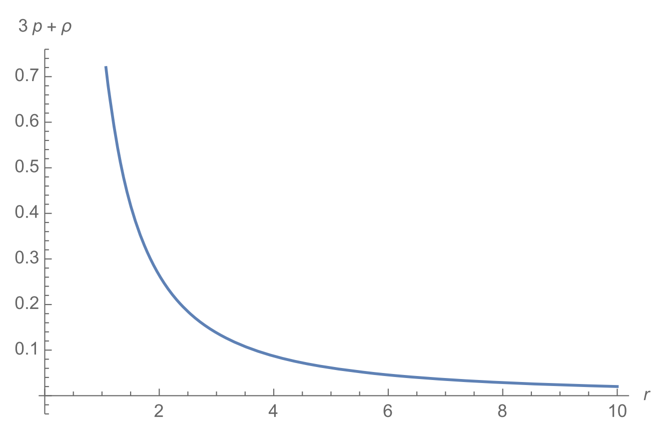

FIG. 5. The plot of $\rho + 3P$ is satisfied for $r_0 = 1$, $l = -2$ and $\kappa = 1$.

V. DEFLECTION OF LIGHT

In this section, we shall explore the effect of bumblebee gravity in the gravitational lensing of the spacetime of the wormhole metric described by Eqs. (3.1) and (3.7). For simplicity, the Lagrangian is chosen in the equatorial plane. Thus, we get

$$2\mathcal{L} = -\dot{t}^2 + \frac{(1+l)\dot{r}^2}{1-(\frac{r_0}{r})^{1+\frac{5wl+3l+3w+1}{wl-l+3w+1}}} + r^2\dot{\varphi}^2. \tag{5.1}$$

There are two constants of motion (energy and angular momentum) for a massless particle, which are defined as follows

$$E = -g_{\mu\nu}K^{\mu}U^{\nu} = \frac{dt}{d\lambda}, \tag{5.2}$$

$$L = g_{\mu\nu}\Phi^{\mu}U^{\nu} = r^2\frac{d\varphi}{d\lambda}, \tag{5.3}$$

in which λ denotes the affine parameter along the light ray. Note that K^{μ} and Φ^{μ} are the timelike and rotational Killing vectors, respectively. One can define the impact parameter of the light ray as

$$b = \frac{L}{E} = r^2\frac{d\varphi}{dt}. \tag{5.4}$$

From the above relations, one can find the following differential equation for the light ray

$$\left(\frac{dr}{d\varphi}\right)^2 + \frac{r^2}{B(r)} = \frac{r^4}{b^2B(r)}, \tag{5.5}$$

in which

$$B(r) = \frac{(1+l)}{1-(\frac{r_0}{r})^{1+\frac{5wl+3l+3w+1}{wl-l+3w+1}}}. \tag{5.6}$$

One can solve this equation by introducing a new variable, let us say $u(\varphi)$, which is related with the radial coordinate as $r = \frac{1}{u(\varphi)}$. If we use the following identity:

$$\frac{\dot{r}}{\dot{\varphi}} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}, \tag{5.7}$$

then in the large r limit, it is possible to recover the following relation

$$\frac{d^2u}{d\varphi^2} + \beta u = 0. \tag{5.8}$$

Furthermore, $\beta = (1+l)^{-1}$, since in the weak limit we have the following approximation: $B(r) \to 1+l$ as $r \to \infty$. The solution of the last differential equation is given by

$$u(\varphi) = A_1\sin(\sqrt{\beta}\varphi) + A_2\cos(\sqrt{\beta}\varphi). \tag{5.9}$$

When using the following initial conditions $u(\varphi = 0) = 0$ and $u(\varphi = \pi/2) = \frac{1}{b}$, we find

$$u(\varphi) = \frac{\sin(\sqrt{\beta}\varphi)}{b}\left(\sin\left(\frac{\sqrt{\beta}\pi}{2}\right)\right)^{-1}. \tag{5.10}$$

Moreover, one can use $\sin\left(\frac{\sqrt{\beta}\pi}{2}\right) \simeq 1$ and in sequel derives the light ray expression:

$$r = \frac{b}{\sin(\sqrt{\beta}\varphi)}. \tag{5.11}$$

This equation is important in computing the deflection angle in the GBT. Next, let us find the wormhole optical metric by letting $\mathrm{d}s^2 = 0$, which corresponds to

$$\mathrm{d}t^2 = \frac{(1+l)dr^2}{1-(\frac{r_0}{r})^{1+\frac{5wl+3l+3w+1}{wl-l+3w+1}}} + r^2\mathrm{d}\varphi^2. \tag{5.12}$$

It is also possible to write down the wormhole optical metric in terms of new coordinates x^a:

$$\mathrm{d}t^2 = h_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\zeta^2 + \mathcal{H}^2(\zeta)\mathrm{d}\varphi^2, \tag{5.13}$$

where

$$\mathrm{d}\zeta = \frac{\sqrt{1+l}\mathrm{d}r}{\sqrt{1-(\frac{r_0}{r})^{1+\frac{5wl+3l+3w+1}{wl-l+3w+1}}}}, \qquad \mathcal{H} = r. \tag{5.14}$$

The Gaussian optical curvature (GOC) $\mathcal{K}$ can be found to be (see for details, [76])

$$\mathcal{K} = -\frac{1}{\mathcal{H}}\left[\frac{\mathrm{d}r}{\mathrm{d}\zeta}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}\zeta}\right)\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}r} + \left(\frac{\mathrm{d}r}{\mathrm{d}\zeta}\right)^2\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}r^2}\right] = -\frac{(1+\Xi)}{2r^2(1+l)}\left(\frac{r_0}{r}\right)^{1+\Xi}, \tag{5.15}$$

where

$$\Xi = \frac{5wl+3l+3w+1}{wl-l+3w+1}. \tag{5.16}$$

Alternatively, one can approximate the above equation by expanding in series around l. Thus, we get

$$\mathcal{K} \simeq -\frac{r_0^2}{r^4} - \frac{r_0^2 l[4\ln(\frac{r_0}{r})(w+1) - w - 1]}{r^4(3w+1)}. \tag{5.17}$$

The key point in this method is that a nonsingular domain outside the light ray, say $\mathcal{A}_R$, which is bounded by $\partial\mathcal{A}_R = \gamma_h \cup C_R$ should be chosen. The GBT in the context of the optical geometry is expressed as follows

$$\iint_{\mathcal{A}_R} \mathcal{K} \mathrm{d}\sigma + \oint_{\partial \mathcal{A}_R} \kappa \mathrm{d}t + \sum_k \psi_k = 2\pi\chi(\mathcal{A}_R), \quad (5.18)$$

in which κ gives the geodesic curvature, $\mathrm{d}\sigma$ is the optical surface element, and ψ_k stands for the exterior angle at the k^{th} vertex. We set the Euler characteristic number to one, i.e., $\chi(\mathcal{A}_R) = 1$. Thus, the geodesic curvature is defined by [76]

$$\kappa = h(\nabla_{\dot{\gamma}} \dot{\gamma}, \ddot{\gamma}), \quad (5.19)$$

where the unit speed condition is selected as $h(\dot{\gamma}, \dot{\gamma}) = 1$. For a very large radial coordinate $R \to \infty$, our two jump angles (at the source $\mathcal{S}$, and observer $\mathcal{O}$), yield $\psi_O + \psi_S \to \pi$ [76]. Thus, the GBT simplifies to

$$\iint_{\mathcal{A}_R} \mathcal{K} \mathrm{d}\sigma + \oint_{C_R} \kappa \mathrm{d}t \overset{R\to\infty}{=} \iint_{\mathcal{A}_\infty} \mathcal{K} \mathrm{d}\sigma + \int_0^{\pi+\hat{\alpha}} \mathrm{d}\varphi = \pi. \quad (5.20)$$

By definition, the geodesic curvature for γ_h is set to zero. Then, we are left with a contribution from the curve C_R, which is located at a distance R from the wormhole center in the equatorial plane. In short, we need to compute the following:

$$\kappa(C_R) = \left|\nabla_{\dot{C}_R} \dot{C}_R\right|. \quad (5.21)$$

In component notation, the radial part can be written as

$$(\nabla_{\dot{C}_R} \dot{C}_R)^r = \dot{C}_R^\varphi (\partial_\varphi \dot{C}_R^r) + \Gamma^{r(op)}_{\varphi\varphi} (\dot{C}_R^\varphi)^2. \quad (5.22)$$

With the help of the unit speed condition, one can calculate the Christoffel symbols that are related to our optical metric in the large coordinate radius R and gets

$$\lim_{R\to\infty} \kappa(C_R) = \lim_{R\to\infty} |\nabla_{\dot{C}_R} \dot{C}_R|, \\ \to \frac{1}{\sqrt{1+l}R}. \quad (5.23)$$

To understand the meaning of the above equation, we rewrite the optical metric for a constant R. Thus, we have

$$\lim_{R\to\infty} \mathrm{d}t \to R \mathrm{d}\varphi. \quad (5.24)$$

Combining the last two equations together, we obtain

$$\kappa(C_R)\mathrm{d}t = \frac{1}{\sqrt{1+l}} \mathrm{d}\varphi. \quad (5.25)$$

This equation implies that our wormhole geometry is nonasymptotically flat and correspondingly, the optical metric is not asymptotically Euclidean. Using this result, we can express the deflection angle as follows

$$\hat{\alpha} = (\sqrt{1+l} - 1)\pi - \sqrt{1+l} \int_0^\pi \int_{\frac{b}{\sin(\frac{\varphi}{\sqrt{1+l}})}}^\infty \mathcal{K} \mathrm{d}\sigma, \quad (5.26)$$

where the light ray $r(\varphi) = \frac{b}{\sin(\frac{\varphi}{\sqrt{1+l}})}$ (b is now interpreted as the *impact parameter* [85]) can be approximated to the closest distance that is obtained from the wormhole in the first order approximation. The first term of Eq. (5.26) can be approximated as

$$(\sqrt{1+l} - 1)\pi = \frac{l\pi}{2} - \frac{l^2\pi}{8} + \cdots \quad (5.27)$$

The surface can also be approximated to

$$\mathrm{d}\sigma = \sqrt{h} \mathrm{d}\zeta \mathrm{d}\varphi \simeq \sqrt{1+l} r \mathrm{d}r \mathrm{d}\varphi. \quad (5.28)$$

Finally, the total deflection angle is found to be

$$\hat{\alpha} \simeq \frac{l\pi}{2} - \int_0^\pi \int_{\frac{b}{\sin(\frac{\varphi}{\sqrt{1+l}})}}^\infty \left[-\frac{(1+\Xi)}{2r}\left(\frac{r_0}{r}\right)^{1+\Xi}\right] \mathrm{d}r \mathrm{d}\varphi. \quad (5.29)$$

Evaluating the last integral, we find

$$\hat{\alpha} \simeq \frac{l\pi}{2} + \frac{\sqrt{(1+l)\pi}}{2}\left(\frac{r_0}{b}\right)^{1+\Xi} \frac{\Gamma(\frac{2+\Xi}{2})}{\Gamma(\frac{3+\Xi}{2})}, \quad (5.30)$$

with the condition of $1+\Xi > 0$. Performing a series expansion, we can write the last equation as follows

$$\hat{\alpha} \simeq \frac{l\pi}{2} + \frac{\pi r_0^2}{4b^2} + \frac{5\pi r_0^2 l}{8b^2} - \frac{\pi r_0^2 l w}{b^2} + \frac{\pi r_0^2 l \ln(\frac{r_0}{b})}{b^2} \\ - \frac{\pi r_0^2 l \ln 2}{b^2} + \frac{2\pi r_0^2 l w \ln 2}{b^2} - \frac{2\pi r_0^2 l w \ln(\frac{r_0}{b})}{b^2}. \quad (5.31)$$

Note that for vanishing bumblebee gravity, $l = 0$, it reduces to the original Einstein's gravity and whence the deflection angle becomes $\hat{\alpha} \simeq \frac{\pi r_0{}^2}{4b^2}$, which is in agreement with the Ellis wormhole [81]. On the other hand, if $1+\Xi \leq 0$, we can only incorporate the finite distance corrections in the deflection angle of light.

It is worthwhile to reemphasize that due to the LSB effect, there are additional terms seen in the left-hand side of Eq. (3.11) that yields a nonasymptotically flat spacetime. Although the first term of Eq. (3.11) is independent from the radial coordinate, however the second term should be vanished when $r \to \infty$, which is achieved by the condition of $1 + \frac{-5l-3}{3l+1} > 0$. Otherwise, the second term blows up, then the GBT becomes problematic. In fact, following paper of Ishihara *et al.* [92], one can only apply a finite correction to the deflection of light. Furthermore, there is a reference paper [85] in which a similar work was carried out for a different spacetime. To conclude, in a certain framework, the GBT can be applied to the spacetimes in the presence of LSB effects.

VI. CONCLUSION

We searched for a way to construct a traversable wormhole solution, one which satisfies the energy conditions to become the most interesting application of the general relativity theory. In finding some realistic matter source that keeps the wormhole throat open such that interstellar or interuniverse travel might become possible, the modified theories of gravity are thought to be new remedy. We have, therefore, considered the bumblebee gravity model to have such a traversable wormhole solution that satisfies the null, weak, and strong energy conditions. In this paper, we first derived the modified Einstein's field equations for the Lorentzian wormhole in the bumblebee gravitational field. Next, using the associated field equations with bumblebee gravity, we have obtained the new traversable wormhole solution with the exact shape function (3.9) and with $w = -\frac{l+1}{5l+3}$. Then, physical features of the obtained wormhole were studied in detail. Singularity of the solution was analyzed by computing the Ricci and Kretschmann scalars. It is seen that the singularity appears when $l = -\frac{1}{3}$. Afterwards, we have checked the flare-out conditions $W' < 1$ for the obtained bumblebee wormhole solution. We have shown that the flare-out conditions are satisfied if $\frac{3l+3}{3l+1} < 1$ where it is plotted in Fig. 2.

In Sec. IV, we checked the energy conditions (null, weak, and strong) for the bumblebee wormhole and rendered them graphically. In Figs. 3–5, we analyzed the three energy conditions of the bumblebee wormhole for the values of $r_0 = 1$, $l = -2$ and $\kappa = 1$. We noted that all energy conditions for the bumblebee wormhole were satisfied when $r_0 = 1$, $l = -2$ and $\kappa = 1$. We also plotted the energy conditions manipulated as interactive in [107–109].

Another important point is that under the LSB effect, the global topology of the wormhole spacetime changes. The limit of $\frac{W(r)}{r}$ at spatial infinity was found to be $\frac{W(r)}{r}\Big|_{r\to\infty} \to \frac{l}{l+1}$, which shows that our wormhole solution was non-asymptotically flat. The deflection of light was computed by applying the GBT to the bumblebee wormhole expressed in the optical geometry. It was seen that bumblebee parameter affects the geodesic optical curvature, modifying the final result for the deflection angle. Due to the nontrivial global topology, we have shown that the deflection of light is changed by $\delta\hat{\alpha} = l\pi/2$, which is purely a topological term and independent of the impact factor b. In addition, we incorporated the bumblebee effects in the total deflection angle by deriving the light ray equation, which modifies the straight line approximation in the domain of integration. In other words, the total deflection angle not only depends on the geometry of the bumblebee traversable wormhole (i.e., throat radius r_0), but it varies with the coupling constant l, and state parameter w. Finally, in the case of $l = 0$, we recovered the Ellis wormhole deflection angle as being reported in the literature [81].

In short, the bumblebee wormhole that we constructed satisfies the energy conditions for normal matter and flare-out conditions near the throat. In the near future, we plan to add new sources (scalar, electromagnetic etc.) to the bumblebee gravity. In this manner, we wish to obtain new spacetime solutions and analyze their physical features [110].

ACKNOWLEDGMENTS

We are thankful to the Editor and anonymous Referees for their constructive suggestions and comments. A. Ö. is grateful to Institute for Advanced Study, Princeton for hospitality. A. Ö. thanks to Prof. Eduardo Guendelman for valuable discussions. This work was supported by the Chilean FONDECYT Grant No. 3170035 (A. Ö.).

APPENDIX: ABOUT EINSTEIN FIELD EQUATIONS OF THE BUMBLEBEE WORMHOLE

The generic line-element of the static and a spherically symmetric traversable wormhole can be expressed as follows

$$ds^2 = -dt^2 + e^{2\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \text{(A1)}$$

which has following nonzero Ricci tensors:

$$R_{rr} = \frac{2\nu'}{r}, \quad \text{(A2)}$$

$$R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = 1 + \frac{r\nu' - 1}{\mathrm{e}^{2\nu(r)}}. \quad \text{(A3)}$$

Recall that the prime symbol denotes the derivative with respect to variable r. Setting

$$b_\mu = [0, b(r), 0, 0], \quad \text{(A4)}$$

and making straightforward calculations, one can obtain the following results from Eq. (2.10):

$$E_{tt}^{\text{instein}} = (3\omega + 1)\rho r\kappa \mathrm{e}^{4\nu(r)} - 2b(r)b'\xi - \psi(r)\xi r, \quad \text{(A5)}$$

$$E_{rr}^{\text{instein}} = r\rho\kappa(\omega - 1)\mathrm{e}^{4\nu(r)} + 4\mathrm{e}^{2\nu(r)}\nu' - 6b(r)b' + 12b(r)^2\nu'\xi - 3\psi(r)\xi, \quad \text{(A6)}$$

$$E_{\theta\theta}^{\text{instein}} \equiv E_{\varphi\varphi}^{\text{instein}} = (2 + \rho\kappa(\omega - 1)r^2)\mathrm{e}^{4\nu(r)} + 2(r\nu' - 1)\mathrm{e}^{2\nu(r)} - 2(-3b(r)^2\nu' r + 3b(r)b' r + b(r)^2)\xi + r\psi(r)\xi, \quad \text{(A7)}$$

where

$$\psi(r)=[3b(r)^2(\nu')^2-b(r)^2\nu''-5b(r)\nu'b'+b(r)b''+(b')^2]r. \tag{A8}$$

One can immediately check that for the traversable wormhole metric (3.1) with the bumblebee vector (3.2):

$$\nu(r)=-\frac{1}{2}\ln\left(1-\frac{W(r)}{r}\right), \tag{A9}$$

$$b(r)=\sqrt{\frac{\alpha}{1-\frac{W(r)}{r}}}, \tag{A10}$$

Eq. (A8) yields $\psi(r)=0$. For this reason, in the obtained Einstein field equations (3.3)–(3.5), there are not any W'^2 and W'' terms.

[1] L. Flamm, Phys. Z. **17**, 48 (1916).
[2] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189 (1916).
[3] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[4] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[5] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[6] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[7] C. Maccone, J. Br. Interplanet. Soc. **48**, 453 (1995).
[8] M. Visser, Phys. Rev. D **41**, 1116 (1990).
[9] H. G. B. Casimir, Proc. Con. Ned. Akad. van Wetensch B **51**, 793 (1948).
[10] P. Kanti, B. Kleihaus, and J. Kunz, Phys. Rev. Lett. **107**, 271101 (2011).
[11] D. Hochberg, A. Popov, and S. V. Sushkov, Phys. Rev. Lett. **78**, 2050 (1997).
[12] T. Harko, F. S. N. Lobo, M. K. Mak, and S. V. Sushkov, Mod. Phys. Lett. A **30**, 1550190 (2015).
[13] J. P. S. Lemos, F. S. N. Lobo, and S. Quinet de Oliveira, Phys. Rev. D **68**, 064004 (2003).
[14] F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005).
[15] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **80**, 104012 (2009).
[16] C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **85**, 044033 (2012).
[17] M. Kord Zangeneh, F. S. N. Lobo, and M. H. Dehghani, Phys. Rev. D **92**, 124049 (2015).
[18] G. Clement, Gen. Relativ. Gravit. **16**, 131 (1984).
[19] G. Clement, Phys. Rev. D **50**, R7119 (1994).
[20] L. Susskind and Y. Zhao, Phys. Rev. D **98**, 046016 (2018).
[21] J.C. Baez and J. Vicary, Classical Quantum Gravity **31**, 214007 (2014).
[22] J. Maldacena and L. Susskind, Fortschr. Phys. **61**, 781 (2013).
[23] E. Poisson and M. Visser, Phys. Rev. D **52**, 7318 (1995).
[24] D. Hochberg and M. Visser, Phys. Rev. D **56**, 4745 (1997).
[25] K. A. Bronnikov and S. W. Kim, Phys. Rev. D **67**, 064027 (2003).
[26] K. A. Bronnikov and J. P. S. Lemos, Phys. Rev. D **79**, 104019 (2009).
[27] K. A. Bronnikov, M. V. Skvortsova, and A. A. Starobinsky, Gravitation Cosmol. **16**, 216 (2010).
[28] R. Garattini and F. S. N. Lobo, Classical Quantum Gravity **24**, 2401 (2007).
[29] M. Cataldo, L. Liempi, and P. Rodriguez, Eur. Phys. J. C **77**, 748 (2017).
[30] M. Cataldo, L. Liempi, and P. Rodriguez, Phys. Lett. B **757**, 130 (2016).
[31] M. Cataldo, L. Liempi, and P. Rodriguez, Phys. Rev. D **91**, 124039 (2015).
[32] M. Cataldo and F. Orellana, Phys. Rev. D **96**, 064022 (2017).
[33] A. Övgün, arXiv:1803.04256.
[34] M.G. Richarte, I.G. Salako, J.P. Morais Graca, H. Moradpour, and A. Övgün, Phys. Rev. D **96**, 084022 (2017).
[35] A. Övgün and K. Jusufi, Eur. Phys. J. Plus **132**, 543 (2017).
[36] M. Halilsoy, A. Ovgun, and S. H. Mazharimousavi, Eur. Phys. J. C **74**, 2796 (2014).
[37] I. Sakalli and A. Ovgun, Eur. Phys. J. Plus **130**, 110 (2015).
[38] I. Sakalli and A. Ovgun, Astrophys. Space Sci. **359**, 32 (2015).
[39] A. Övgün and I. Sakalli, Theor. Math. Phys. **190**, 120 (2017).
[40] A. Ovgun and M. Halilsoy, Astrophys. Space Sci. **361**, 214 (2016).
[41] A. Ovgun, Eur. Phys. J. Plus **131**, 389 (2016).
[42] A. Övgün and K. Jusufi, Adv. High Energy Phys. **2017**, 1215254 (2017).
[43] S. W. Kim, Phys. Lett. A **166**, 13 (1992).
[44] S. V. Sushkov and S. W. Kim, Classical Quantum Gravity **19**, 4909 (2002).
[45] S. W. Kim and H. Lee, Phys. Rev. D **63**, 064014 (2001).
[46] E. Guendelman, A. Kaganovich, E. Nissimov, and S. Pacheva, Int. J. Mod. Phys. A **26**, 5211 (2011).

[47] E. I. Guendelman, Int. J. Mod. Phys. D **19**, 1357 (2010).
[48] E. I. Guendelman, A. Kaganovich, E. Nissimov, and S. Pacheva, Fortschr. Phys. **57**, 566 (2009).
[49] E. I. Guendelman, A. Kaganovich, E. Nissimov, and S. Pacheva, Int. J. Mod. Phys. A **25**, 1405 (2010).
[50] E. I. Guendelman, Gen. Relativ. Gravit. **23**, 1415 (1991).
[51] N. Montelongo Garcia and F. S. N. Lobo, Classical Quantum Gravity **28**, 085018 (2011).
[52] P. Bhar, F. Rahaman, T. Manna, and A. Banerjee, Eur. Phys. J. C **76**, 708 (2016).
[53] F. Rahaman, N. Paul, A. Banerjee, S. S. De, S. Ray, and A. A. Usmani, Eur. Phys. J. C **76**, 246 (2016).
[54] P. Bhar and F. Rahaman, Eur. Phys. J. C **74**, 3213 (2014).
[55] F. Rahaman, P. Salucci, P. K. F. Kuhfittig, S. Ray, and M. Rahaman, Ann. Phys. (Amsterdam) **350**, 561 (2014).
[56] F. Rahaman, I. Karar, S. Karmakar, and S. Ray, Phys. Lett. B **746**, 73 (2015).
[57] P. K. F. Kuhfittig, Eur. Phys. J. C **74**, 2818 (2014).
[58] F. Rahaman, P. K. F. Kuhfittig, S. Ray, and N. Islam, Eur. Phys. J. C **74**, 2750 (2014).
[59] M. Jamil, F. Rahaman, R. Myrzakulov, P. K. F. Kuhfittig, N. Ahmed, and U. F. Mondal, J. Korean Phys. Soc. **65**, 917 (2014).
[60] F. Rahaman, S. Islam, P. K. F. Kuhfittig, and S. Ray, Phys. Rev. D **86**, 106010 (2012).
[61] R. Bluhm and V. A. Kostelecky, Phys. Rev. D **71**, 065008 (2005).
[62] O. Bertolami and J. Paramos, Phys. Rev. D **72**, 044001 (2005).
[63] R. Bluhm, N. L. Gagne, R. Potting, and A. Vrublevskis, Phys. Rev. D **77**, 125007 (2008); **79**, 029902(E) (2009).
[64] M. D. Seifert, Phys. Rev. D **81**, 065010 (2010).
[65] A. V. Kostelecky and J. D. Tasson, Phys. Rev. D **83**, 016013 (2011).
[66] C. A. Escobar and A. Martin-Ruiz, Phys. Rev. D **95**, 095006 (2017).
[67] M. H. Dickinson, F. O. Lehmann, and S. P. Sane, Science **284**, 1954 (1999).
[68] V. A. Kostelecky and S. Samuel, Phys. Rev. D **40**, 1886 (1989).
[69] R. Casana, A. Cavalcante, F. P. Poulis, and E. B. Santos, Phys. Rev. D **97**, 104001 (2018).
[70] O. Bertolami and C. Carvalho, J. Phys. Conf. Ser. **67**, 012009 (2007).
[71] R. V. Maluf, J. E. G. Silva, and C. A. S. Almeida, Phys. Lett. B **749**, 304 (2015).
[72] D. Capelo and J. Paramos, Phys. Rev. D **91**, 104007 (2015).
[73] A. F. Santos, A. Y. Petrov, W. D. R. Jesus, and J. R. Nascimento, Mod. Phys. Lett. A **30**, 1550011 (2015).
[74] R. V. Maluf, C. A. S. Almeida, R. Casana, and M. M. Ferreira, Jr., Phys. Rev. D **90**, 025007 (2014).
[75] J. Paramos and G. Guiomar, Phys. Rev. D **90**, 082002 (2014).
[76] G. W. Gibbons and M. C. Werner, Classical Quantum Gravity **25**, 235009 (2008).
[77] M. C. Werner, Gen. Relativ. Gravit. **44**, 3047 (2012).
[78] C. F. Gauss, *Werke* (K. Gesellschaft Wissenschaft, Göttingen, 1900), Vol. 8; O. Bonnet, J. Éc. Polytech. **19**, 1 (1848).
[79] G. W. Gibbons, C. A. R. Herdeiro, C. Warnick, and M. C. Werner, Phys. Rev. D **79**, 044022 (2009).
[80] K. Jusufi, Astrophys. Space Sci. **361**, 24 (2016).
[81] K. Jusufi, Int. J. Geom. Methods Mod. Phys. **14**, 1750179 (2017).
[82] K. Jusufi and A. Övgün, Phys. Rev. D **97**, 024042 (2018).
[83] K. Jusufi, M. C. Werner, A. Banerjee, and A. Övgün, Phys. Rev. D **95**, 104012 (2017).
[84] K. Jusufi, A. Övgün, and A. Banerjee, Phys. Rev. D **96**, 084036 (2017).
[85] K. Jusufi, I. Sakalli, and A. Övgün, Phys. Rev. D **96**, 024040 (2017).
[86] K. Jusufi, N. Sarkar, F. Rahaman, A. Banerjee, and S. Hansraj, Eur. Phys. J. C **78**, 349 (2018).
[87] K. Jusufi, F. Rahaman, and A. Banerjee, Ann. Phys. (Amsterdam) **389**, 219 (2018).
[88] I. Sakalli and A. Övgün, Europhys. Lett. **118**, 60006 (2017).
[89] A. Övgün, I. Sakalli, and J. Saavedra, J. Cosmol. Astropart. Phys. 10 (2018) 041.
[90] A. Övgün, K. Jusufi, and I. Sakalli, Ann. Phys. (Amsterdam) **399**, 193 (2018).
[91] A. Övgün, Phys. Rev. D **98**, 044033 (2018).
[92] A. Ishihara, Y. Suzuki, T. Ono, and H. Asada, Phys. Rev. D **95**, 044017 (2017).
[93] T. Ono, A. Ishihara, and H. Asada, Phys. Rev. D **96**, 104037 (2017).
[94] G. Crisnejo and E. Gallo, Phys. Rev. D **97**, 124016 (2018).
[95] K. Nakajima and H. Asada, Phys. Rev. D **85**, 107501 (2012).
[96] N. Tsukamoto, T. Harada, and K. Yajima, Phys. Rev. D **86**, 104062 (2012).
[97] R. Shaikh, Phys. Rev. D **98**, 024044 (2018).
[98] R. Shaikh and S. Kar, Phys. Rev. D **94**, 024011 (2016).
[99] R. Shaikh, Phys. Rev. D **92**, 024015 (2015).
[100] V. Alan Kostelecky, Phys. Rev. D **69**, 105009 (2004).
[101] O. Bertolami and J. Paramos, Phys. Rev. D **72**, 044001 (2005).
[102] K. Jusufi, Phys. Rev. D **98**, 044016 (2018).
[103] G. Clement and C. Leygnac, Phys. Rev. D **70**, 084018 (2004).
[104] S. S. Yazadjiev, Classical Quantum Gravity **22**, 3875 (2005).
[105] T. Muller, Phys. Rev. D **77**, 044043 (2008).
[106] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[107] Interactive plot of $\rho + p$ of bumblebee wormhole, https://www.wolframcloud.com/objects/b9c3ae67-7f31-466f-ae9d-7e2c1f08139f.
[108] Interactive plot of $\rho + 3p$ of bumblebee wormhole, https://www.wolframcloud.com/objects/6bd52e2c-f49c-41b2-8d28-e2153c4c10b1.
[109] Interactive plot of energy density ρ of bumblebee wormhole, https://www.wolframcloud.com/objects/72a4aa0e-8fac-4a66-a07d-e4cc53333ded.
[110] Github page of bumblebee model of gravity project, https://bumblebeegr.github.io/BIAS/.