

Shadow cast and deflection angle of Kerr-Newman-Kasuya spacetime

To cite this article: Ali Övgün *et al* JCAP10(2018)041

View the [article online](#) for updates and enhancements.



AAS | **IOP Astronomy** ebooks

Part of your publishing universe and your first choice for astronomy, astrophysics, solar physics and planetary science ebooks.

iopscience.org/books/aas

Shadow cast and deflection angle of Kerr-Newman-Kasuya spacetime

Ali Övgün,^{a,b,c,1} İzzet Sakallı^b and Joel Saavedra^a

^aInstituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile

^bPhysics Department, Arts and Sciences Faculty, Eastern Mediterranean University, Famagusta, North Cyprus via Mersin 10, Turkey

^cSchool of Natural Sciences, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, U.S.A.

E-mail: ali.ovgun@pucv.cl, izzet.sakalli@emu.edu.tr, joel.saavedra@ucv.cl

Received July 3, 2018

Revised September 30, 2018

Accepted October 11, 2018

Published October 22, 2018

Abstract. We study the shadow cast or silhouette generated by a Kerr-Newman-Kasuya (KNK) spacetime (rotating dyon black hole). It is shown that in addition to the angular momentum of the black hole, the dyon charge also affects the shadow image of the KNK black hole. Moreover, we analyze the weak gravitational lensing by the KNK black hole by using the Gauss-Bonnet theorem. Finally, we find that extra dyon charge decreases both the deflection angle and shadow of the KNK black hole.

Keywords: GR black holes, gravitational lensing, gravity, gravitational waves / theory

ArXiv ePrint: [1807.00388](https://arxiv.org/abs/1807.00388)

¹Corresponding author.

Contents

1	Introduction	1
2	KNK spacetime	2
3	Deflection angle of light by KNK spacetime	4
4	Shadows of KNK spacetime	5
5	Conclusion	11

1 Introduction

Black holes are the most interesting objects in the universe. Black hole has a event horizon, as a boundary where nothing can escape include the light so that they are called as black [1]. But beyond that line, particles can escape. Black hole always try to pull the surrounding matter which is known as accretion [2, 3]. During the accretion of black holes, they release large amounts of energy into their large-scale environments, so that black holes may play a prominent role in the processes that control galaxy formation [4, 5]. Moreover, this accreting matter heats up through viscous dissipation and radiate light in various frequencies. For instance, the radio waves are one of them which can be detected through the radio telescopes [6–8]. When the accretion happens onto black hole, shining material pass through the event horizon, which results in a dark area over a bright background: black hole shadow (BHS) [9]. We can say that this shadow is actually an image of the event horizon. The center of galaxies is a playground of a gigantic black holes. Because of the gravitational lens effect, the background would have cast a shade larger than its horizon size [10]. The size and shape of this shadow can be calculated and visualized, respectively.

In 1970s, Bardeen, Press and Teukolsky [11] and then Chandrasekhar [12] computed that a BHS has a radius of $r_{\text{shadow}} = \sqrt{27}M = 5.2M$ over the background light source (seen by an outside observer). As reported by many numerical calculations, rotating black holes cast shadows of approximately the same size as well [13–64]. On the other hand, null geodesics method and gravitational lensing are also useful tools to gain information about the black holes [65, 66]. Strong and weak gravitational lensing by black holes, wormholes or other exotic objects have been investigated by several authors [67–71].

The main purpose of studying the gravitational lensing is to detect the black holes in the universe, which are believed that they are mostly located at the center of the galaxies. The strong gravitational lensing helps us to find the position, magnification, and time delays of the images by black holes. Moreover, in “weak lensing” the effect is much weaker but can still be detected statistically [72]. To do so, Gibbons and Werner generated a new technique to calculate the deflection angle of light rays within the asymptotic source and receiver [73]. Their method is based on the famous Gauss-Bonnet theorem (GBT), which solves the integral in an infinite domain bounded by the light ray. After locating the source and receiver to the asymptotic Minkowski regions, the deflection angle of optical metric of a static black hole was computed [74]. Then Werner extended to the stationary black holes by employing the Finsler-Randers type optical geometry with Nazım’s osculating Riemannian manifold [75]. In the

sequel, this topic has been thoroughly studied by several authors for different types of spacetimes [76–92]. Recently, A. Ishihara et al. have shown that for the spherical symmetric static objects, it is possible to find the deflection angle by considering the finite-distance corrections instead of using the asymptotic receiver and source [93, 94]. Then, T. Ono et al. have extended the method for the axisymmetric spacetimes [93–95]. The expectation of the detection of the Milky Way’s central supermassive black hole (Sagittarius A* is the site of that black hole) by the Event Horizon Telescope (EHT) increases the impact on the studies about the BHS. EHT tries to observe the extremely hot gas around the event horizon of the black hole [96].

It is believed that there would exist formation (and creation) mechanism of gravitomagnetic charge in the gravitational interaction, just as some prevalent theories [97] that provide the theoretical mechanism of existence of magnetic monopole in various gauge interactions. Magnetic monopole in electrodynamics and gauge field theory has been extensively discussed and sought after for decades, and the existence of the ’t Hooft-Polyakov monopole solution has spurred new interest of both theorists and experimentalists [98, 99]. The gravitomagnetic charge is the proposed gravitational analogue of Dirac’s magnetic monopole [100]. However, if it is indeed present in universe, it will also lead to significant consequences in astrophysics and cosmology. On the other hand, as it is well-known, dyon is a pole possessing both electric and magnetic charges. In the “weak field limit”, Einstein’s equations reduce to a form remarkably like Maxwell’s equations of electromagnetism. Terms appear that are analogous to the electric field caused by charges and the magnetic field produced by the flow of charge. The “electric terms” correspond simply to the gravity that keeps our feet on the ground. The “magnetic terms” are wholly unfamiliar; we do not see them in everyday life. The KNK spacetime [101] is nothing but a rotating dyon (a hypothetical particle in 4-dimensional theories with both electric and magnetic charges) black hole. The best place to measure gravitomagnetism is in Earth orbit. Just as a spinning ball of electric charge produces a well-defined magnetic field, a spinning mass such as Earth is expected to produce a well-defined gravitomagnetic field. On the other hand, recent studies of gauge theories have given promising results about the existence of the monopole [102, 103]. Besides, there is a renewed interest for the cosmological constant since it can be one of the theoretical models to explain the inflationary scenario of the early universe. In this scenario, the universe undergoes a stage which is geometrically described by hot de-Sitter(dS) spacetime which is related with KNK spacetime [104]. In addition to this, it was shown that primordial universe can be described by the KNK spacetime [105]. In the present study, we consider the KNK spacetime and its shadow cast. To this end, we employ the GBT and analyze the weak gravitational lensing by the KNK black hole.

This paper is organized as follows. Section 2 briefly describes the KNK black hole and serves its physical features. We compute the deflection angle by following the GBT of [73] in section 3. Section 4 is devoted to the computation of the shadows of the KNK black holes by manipulating the rotation parameter and the dyon charge. Conclusions are presented in section 5.

2 KNK spacetime

The metric of the KNK black hole in the Boyer-Lindquist coordinates is given by [101]

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2)d\phi - a dt]^2, \quad (2.1)$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q_e^2 + Q_m^2, \quad (2.2)$$

and

$$\Sigma = r^2 + a^2 \cos^2 \theta. \quad (2.3)$$

Note that M is the mass, $a = J/M$ represents the rotation parameter, which is the angular momentum per unit mass, Q_e and Q_m denote electric and magnetic charges, respectively. The spacetime of the KNK black hole reduces to the Kerr-Newman black hole when $Q_m = 0$, the Kerr black hole if $Q_e = Q_m = 0$, the Reissner-Nordström black hole for $Q_m = a = 0$ and the Schwarzschild black hole if $a = Q_e = Q_m = 0$. Meanwhile, the KNK spacetime is one of the members of Plebański-Demiański family of black hole solutions [106].

Event horizon of the KNK black hole is obtained by using the following equation

$$\Delta = (r - r_+)(r - r_-) = 0, \quad (2.4)$$

whose solutions are

$$r_+ = M + [M^2 - (a^2 + Q_e^2 + Q_m^2)]^{1/2}, \quad (2.5)$$

and

$$r_- = M - [M^2 - (a^2 + Q_e^2 + Q_m^2)]^{1/2}. \quad (2.6)$$

The surface gravity [107] of the KNK black hole can be obtained as follows:

$$\kappa_+ \equiv \frac{1}{2} \frac{1}{r_+^2 + a^2} \frac{d\Delta}{dr} \Big|_{r=r_+} = \frac{1}{2} \frac{r_+ - r_-}{r_+^2 + a^2}. \quad (2.7)$$

Thus, the Hawking temperature [107] of the KNK black hole becomes

$$T_+ = \frac{\kappa_+}{2\pi} = \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2 + a^2}. \quad (2.8)$$

The surface area of the horizon is given by

$$\mathcal{A}_+ = \int \int \sqrt{-g} \, d\theta \, d\phi \Big|_{r=r_+} = 4\pi(r_+^2 + a^2), \quad (2.9)$$

where $g \equiv \det(g_{\sigma\tau}) = -r^4 \sin^2 \theta$.

The entropy [107] of the KNK black hole at the event horizon reads

$$S_+ = \frac{\mathcal{A}_+}{4} = \pi(r_+^2 + a^2). \quad (2.10)$$

The angular velocity of KNK black hole is given by

$$\Omega_+ = - \frac{g_{t\phi}}{g_{\phi\phi}} \Big|_{r=r_+} = \frac{a}{(r_+^2 + a^2)}. \quad (2.11)$$

Furthermore, the electric potential with magnetic and electric charges is as follows

$$\Phi_+ = \frac{Q_e r_+ - \xi Q_m a}{r_+^2 + a^2}. \quad (2.12)$$

Note that $\xi = \pm 1$ stands for the two gauges. The first law of the thermodynamics is satisfied via the following expression:

$$dM = T_+ \, dS_+ + \Omega_+ \, dJ + \Phi_+ \, dQ_e. \quad (2.13)$$

3 Deflection angle of light by KNK spacetime

In this section, by using the GBT we shall study the deflection angle for the KNK black hole. We use the null condition $ds^2 = 0$ and solve the KNK spacetime for dt as follows:

$$dt = \sqrt{\gamma_{ij}dx^i dx^j} + \beta_i dx^i. \quad (3.1)$$

Note that γ_{ij} goes as $(i, j = 1, 2, 3)$. Then, we obtain the components of the optical metric of KNK spacetime in terms of γ_{ij} and β_i :

$$d\ell^2 = \gamma_{ij}dx^i dx^j = \frac{\Sigma^2}{\Delta(\Sigma - 2Mr)} dr^2 + \frac{\Sigma^2}{(\Sigma - 2Mr)} d\theta^2 + \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{(\Sigma - 2Mr)} \right) \frac{\Sigma \sin^2 \theta}{(\Sigma - 2Mr)} d\phi^2, \quad (3.2)$$

$$\beta_i dx^i = -\frac{2aMr \sin^2 \theta}{(\Sigma - 2Mr)} d\phi. \quad (3.3)$$

The spatial metric γ_{ij} stands for an arc-length (ℓ) along the light ray. At the equatorial plane ($\theta = \pi/2$), we have

$$d\ell^2 = \frac{\Sigma^2}{\Delta(\Sigma - 2Mr)} dr^2 + \left(r^2 + a^2 + \frac{2a^2 Mr}{(\Sigma - 2Mr)} \right) \frac{\Sigma}{(\Sigma - 2Mr)} d\phi^2, \quad (3.4)$$

and

$$\beta_i dx^i = -\omega d\phi, \quad (3.5)$$

where $\omega = \frac{2aMr}{(\Sigma - 2Mr)} d\phi$. In practice, we define the deflection angle ($\hat{\alpha}$) with the angles Ψ_R , Ψ_S and ϕ_{RS} which correspond to receiver, source, and coordinate, respectively. Namely, we have

$$\hat{\alpha} \equiv \Psi_R - \Psi_S + \phi_{RS}. \quad (3.6)$$

Locating the position of the receiver and source to suitable locations where the endpoints of light rays lie in the Euclidean space, the GBT [108] admits the following expression for the deflection angle:

$$\hat{\alpha} = - \iint_{\infty_R \square \infty_S} K dS + \int_S^R \kappa_g d\ell. \quad (3.7)$$

We should note that $\infty_R \square \infty_S$ is embedded quadrilateral. Moreover, κ_g is for the geodesic curvature and $d\ell$ is for an arc length. For evaluating the first integral of eq. (3.7), one should first calculate the Gaussian optical curvature (related with 2-dimensional Riemann tensor) in the weak field approximation:

$$K = \frac{R_{r\phi r\phi}}{\det \gamma} = -2 \frac{M}{r^3} - 3 \frac{(-r + 2M) Q_m^2}{r^5} - 3 \frac{(-r + 2M) Q_e^2}{r^5}. \quad (3.8)$$

Then, we compute the following geodesic curvature [108]:

$$\kappa_g = -\sqrt{\frac{1}{\gamma\gamma^{\theta\theta}}} \beta_{\phi,r}, \quad (3.9)$$

where γ is the $\det \gamma$.

For the KNK black hole, the geodesic curvature κ_g is computed as

$$\kappa_g = -\frac{2aM}{r^3}. \quad (3.10)$$

We now examine the net contribution of the geodesic part:

$$\int_S^R \kappa_g dl = \int_R^S \frac{2aM}{r^3} dl = \frac{4aM}{b^2}, \quad (3.11)$$

where b is the impact parameter. To visualize the boundary of the integration domain, we define the angle by using the outgoing radial direction of the light rays:

$$\sin \Psi = \frac{\sqrt{\beta^2 + r^2}b}{r^2} - \frac{\beta^2 a}{r^4 \sqrt{\beta^2 + r^2}} - \frac{bM}{r \sqrt{\beta^2 + r^2}}, \quad (3.12)$$

and the solution for the photon orbit is found to be

$$u = \frac{\sin \phi}{b} + \frac{M(1 + \cos^2 \phi)}{b^2} - \frac{2aM}{b^3}. \quad (3.13)$$

Afterward, we calculate the integral of the Gaussian curvature of the optical metric of the KNK black hole [74]

$$- \iint_{\infty_R \square \infty_S} K dS = \int_0^\pi \int_\infty^u K dr d\phi = \frac{4M}{b} - \frac{3Q_e^2}{4b^2} - \frac{3Q_m^2}{4b^2}. \quad (3.14)$$

In sequel, we combine the solutions of the Gaussian optical curvature integral and geodesic curvature integral to obtain the total deflection angle of light on the equatorial plane of the KNK black hole:

$$\hat{\alpha} = \frac{4M}{b} - \frac{3Q_e^2}{4b^2} - \frac{3Q_m^2}{4b^2} \pm \frac{4aM}{b^2}. \quad (3.15)$$

We note that the positive sign stands for the retrograde and negative sign is for the prograde case of the photon orbit. The deflection angle of the KNK black hole is agreed with the Kerr case with the limit of $Q_e = Q_m = 0$ [109] and the non-rotating dyon black hole if $a = 0$ [110]. Moreover, the deflection angle of the charged black hole is recovered if $a = Q_m = 0$ and the deflection angle of the Schwarzschild black hole is obtained for $a = Q_e = Q_m = 0$. The deflection angle is linearly decreased with the extra magnetic charge compared to Kerr-Newman black hole.

4 Shadows of KNK spacetime

The KNK spacetime in Boyer-Lindquist coordinates $g_{\mu\nu}^K$ is described by the following metric tensors:

$$\begin{aligned} g_{tt}^{\text{KNK}} &= - \left(1 - \frac{2Mr}{\Sigma} \right), \\ g_{t\phi}^{\text{KNK}} &= - \frac{2Mar \sin^2 \theta}{\Sigma}, \\ g_{rr}^{\text{KNK}} &= \frac{\Sigma}{\Delta}, \\ g_{\theta\theta}^{\text{KNK}} &= \Sigma, \\ g_{\phi\phi}^{\text{KNK}} &= \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma} \right) \sin^2 \theta, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned}\Delta &\equiv r^2 - 2Mr + a^2 + Q_m^2 + Q_e^2, \\ \Sigma &\equiv r^2 + a^2 \cos^2 \theta.\end{aligned}\quad (4.2)$$

The motion of the particle on the KNK is obtained by the following Lagrangian: [12]

$$\mathcal{L} = \frac{1}{2} g_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma, \quad (4.3)$$

where $\dot{x}^\nu = u^\nu = dx^\nu/d\lambda$ in which u^ν stands for four velocity of the particle with the affine parameter λ . Due to the symmetry of black hole, conjugate momenta p_t and p_ϕ are conserved due to the metric-independent variables t and ϕ . Hence, the energy E and angular momentum L are obtained as:

$$E = p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{\phi t} \dot{\phi} + g_{tt} \dot{t}, \quad L = -p_\phi = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -g_{\phi\phi} \dot{\phi} - g_{\phi t} \dot{t}. \quad (4.4)$$

and we derive

$$\Sigma \dot{t} = -a(aE \sin^2 \theta - L) + \frac{(r^2 + a^2)P(r)}{\Delta(r)}, \quad (4.5)$$

$$\Sigma \dot{\phi} = -\left(aE - \frac{L}{\sin^2 \theta}\right) + \frac{aP(r)}{\Delta(r)}, \quad (4.6)$$

where $P(r) \equiv E(r^2 + a^2) - aL$. To calculate the other geodesics equations, we use the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{\nu\sigma} \frac{\partial S}{\partial x^\nu} \frac{\partial S}{\partial x^\sigma},$$

with the following ansatz:

$$S = \frac{1}{2} \mu^2 \lambda - Et + L\phi + S_r(r) + S_\theta(\theta),$$

where μ is proportional to the rest mass of the particle.

For the KNK spacetime, the Hamilton-Jacobi equation yields:

$$\frac{1}{2} g^{tt} \frac{\partial S}{\partial x^t} \frac{\partial S}{\partial x^t} + g^{\phi t} \frac{\partial S}{\partial x^t} \frac{\partial S}{\partial x^\phi} + \frac{1}{2} g^{rr} \frac{\partial S}{\partial x^r} \frac{\partial S}{\partial x^r} + \frac{1}{2} g^{\theta\theta} \frac{\partial S}{\partial x^\theta} \frac{\partial S}{\partial x^\theta} + \frac{1}{2} g^{\phi\phi} \frac{\partial S}{\partial x^\phi} \frac{\partial S}{\partial x^\phi} = -\frac{\partial S}{\partial \lambda}.$$

Then, we solve it for S_r and S_θ as follows [12]:

$$\Sigma \frac{\partial S_r}{\partial r} = \pm \sqrt{R(r)}, \quad (4.7)$$

$$\Sigma \frac{\partial S_\theta}{\partial \theta} = \pm \sqrt{\Theta(\theta)}, \quad (4.8)$$

where

$$R(r) \equiv P(r)^2 - \Delta(r) [(L - aE)^2 + \mathcal{Q}], \quad (4.9)$$

$$\Theta(\theta) \equiv \mathcal{Q} + \cos^2 \theta \left(a^2 E^2 - \frac{L^2}{\sin^2 \theta} \right), \quad (4.10)$$

and \mathcal{Q} is the Carter constant defined by $\mathcal{Q} \equiv \mathcal{K} - (L - aE)^2$ where \mathcal{K} is a constant of motion. $R(r)$ and $\Theta(\theta)$ should be non-negative for the photon motion. Two impact parameters η and ξ are introduced in terms of energy E , angular momentum L and Carter constant \mathcal{Q} as [12]

$$\xi \equiv \frac{L}{E}, \quad \eta \equiv \frac{\mathcal{Q}}{E^2}, \quad (4.11)$$

For photon case, eq. (4.3) can be rewritten in terms of dimensionless quantities η and ξ :

$$R(r) = \frac{1}{E^2} [(r^2 + a^2) - a\xi]^2 - \Delta [(a - \xi)^2 + \eta]. \quad (4.12)$$

Note that R and Θ act as effective potentials for moving particle in r and θ directions, respectively. Equation S_r can be represented as

$$\left(\frac{\partial S_r}{\partial r} \right)^2 + V_{\text{eff}} = 0, \quad (4.13)$$

where V_{eff} is the effective potential:

$$V_{\text{eff}} = \frac{1}{\Sigma^2} [(r^2 + a^2) - a\xi]^2 - \Delta [(a - \xi)^2 + \eta]. \quad (4.14)$$

We can achieve the most critical and unstable circular orbit by maximizing the effective potential, which should satisfy the following conditions

$$V_{\text{eff}} = \frac{\partial V_{\text{eff}}}{\partial r} \Big|_{r=r_0} = 0 \quad \text{or} \quad R = \frac{\partial R}{\partial r} \Big|_{r=r_0} = 0, \quad (4.15)$$

where $r = r_0$ is the radius of the unstable circular null orbit. We assume that the photons and the observer are located at the infinity ($\mu = 0$) and photons come near the equatorial plane ($\theta = \frac{\pi}{2}$). We solve the eq. (4.15) and obtain the following celestial coordinates of the image:

$$\xi = \frac{r^2 - r\Delta - a^2}{a(r-1)}, \quad (4.16)$$

$$\eta = \frac{r^3[4\Delta - r(r-1)^2]}{a^2(r-1)^2}. \quad (4.17)$$

Note that observer at spatial infinity can observe the celestial coordinates of the image and determine the contour of the BHS. We briefly review the expressions of the images of photon rings around black holes. There are three independent constants of motion in the KNK metric. The inertial reference frame of an observer with the basis vectors $\{e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$ and coordinate basis of the metric $\{e_t, e_r, e_\theta, e_\phi\}$ [14]:

$$\begin{aligned} e_{\hat{t}} &= \zeta e_t + \gamma e_\phi, \\ e_{\hat{r}} &= \frac{1}{\sqrt{g_{rr}}} e_r, \\ e_{\hat{\theta}} &= \frac{1}{\sqrt{g_{\theta\theta}}} e_\theta, \\ e_{\hat{\phi}} &= \frac{1}{\sqrt{g_{\phi\phi}}} e_\phi, \end{aligned} \quad (4.18)$$

which are defined, at a large distance from the KNK black hole, with the relation between local observer's basis vector and coordinate basis of metric: $e_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}} e_{\mu}$ with $e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}$ for the Minkowski metric $\eta_{\hat{\alpha}\hat{\beta}} = \text{diag}(-1, 1, 1, 1)$. From the orthonormal property of ζ and γ , one obtains the following equations [11, 35]:

$$\begin{aligned}\zeta &= \sqrt{\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}, \\ \gamma &= -\frac{g_{t\phi}}{g_{\phi\phi}} \sqrt{\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}.\end{aligned}\quad (4.19)$$

Then, the energy and angular momentum are obtained as:

$$p^{\hat{t}} = \zeta E - \gamma L, \quad (4.20)$$

$$p^{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} L. \quad (4.21)$$

On the other hand, the quantities α and β are known as the impact parameters. They are the axes of the Cartesian coordinate of the image plane of the observer located at a distance $r = r_0$ and inclination angle $\theta = i$ between the rotation axis of KNK black hole and observer's line of sight:

$$\alpha \equiv -r_0 \frac{p^{\hat{\phi}}}{p^{\hat{t}}} = -r_0 \frac{\xi}{\sqrt{g_{\phi\phi}} \zeta \left(1 + \frac{g_{t\phi}}{g_{\phi\phi}} \xi\right)}, \quad (4.22)$$

$$\beta \equiv r_0 \frac{p^{\hat{\theta}}}{p^{\hat{t}}} = r_0 \frac{\pm \sqrt{\Theta(i)}}{\sqrt{g_{\theta\theta}} \zeta \left(1 + \frac{g_{t\phi}}{g_{\phi\phi}} \xi\right)}, \quad (4.23)$$

where

$$p^{\hat{\theta}} = \frac{p_{\theta}}{\sqrt{g_{\theta\theta}}} = \frac{\pm \sqrt{\Theta(i)}}{\sqrt{g_{\theta\theta}}}, \quad (4.24)$$

$$\Theta(i) \equiv \eta + a^2 \cos^2 i - \xi^2 \cot^2 i. \quad (4.25)$$

All the metric elements seen in the above expressions are evaluated at $r = r_0$ and $\theta = i$. In the limit $r_0 \rightarrow \infty$, the shape of the BHS for a far away observer can be determined by the following celestial coordinates [11, 12]:

$$\alpha = \lim_{r \rightarrow \infty} \left(-r^2 \sin \theta \frac{d\phi}{dr} \Big|_{\theta \rightarrow i} \right), \quad (4.26)$$

$$\beta = \pm \lim_{r \rightarrow \infty} \left(r^2 \frac{d\theta}{dr} \Big|_{\theta \rightarrow i} \right), \quad (4.27)$$

which yield

$$\alpha = -\xi \csc i, \quad (4.28)$$

$$\beta = \pm \sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i}. \quad (4.29)$$

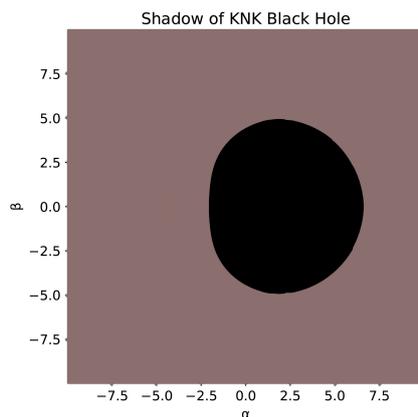


Figure 1. Shadow of KNK black hole for $Q_e = 0.1$, $Q_m = 0.1$, $M = 1$, and $a = 0.99$.

The celestial coordinates α and β show the apparent perpendicular distances of the image around the black hole. For the KNK black hole, at $i = \pi/2$, the celestial coordinates become

$$\alpha = - \left(\frac{r^2 - r\Delta - a^2}{a(r-1)} \right), \quad (4.30)$$

$$\beta = \pm \sqrt{\frac{r^3[4\Delta - r(r-1)^2]}{a^2(r-1)^2}}. \quad (4.31)$$

Figures 1 and 2 show the plots of the shadows of the KNK black hole with different values of spin a and magnetic charge Q_m . Top figures of figure 2 show the Schwarzschild case with dashed lines to compare with KNK black hole. For a fixed a value, the presence of a charge Q_m leads to a smaller shadow than in the case of Kerr geometry, and a value of Q_m gives a more distorted shadow compared to Kerr-Newmann black hole. The shadows of KNK black hole with the different inclination angles are also plotted in figure 3.

The EHT tries to understand the “event horizon” of two galactic center black holes such as Sgr A* black hole in the Milky Way galaxy and supermassive black hole in galaxy M87 which is about 1500 times more massive and 2000 times farther away than Sgr A*. However, the size of the black hole’s event horizon in M87 is smaller than the event horizon of the Sgr A* black hole, on the other hand, it is large enough for the EHT to resolve. For example: M87 black hole is about 22 micro-arc-sec as compared to the 53 micro-arc-sec of Sgr A* black hole so that the EHT requires strong angular resolution to match the small angular size of these black holes. The angular size of the shadow can be calculated

$$\theta_s = R_s M / D_o, \quad (4.32)$$

where R_s is the angular radius, D_o is the distance from the observer to the black hole and M is the mass of the black hole. For the supermassive black hole Sgr A*, we have $M = 4.3 \times 10^6 M_\odot$ and $D_o = 8.3 \text{ kpc}$ [113]; then, for the fixed parameters $Q_e = 0.1$ and $a = 0.9$, we obtain:

Q_m	0.1	0.3	0.5	0.6
$\theta_s (\mu\text{as})$	26.77	25.91	24.29	22.45

Here, the table shows that resolutions are needed to gain information from observations of the shadow of the Sgr A* black hole.

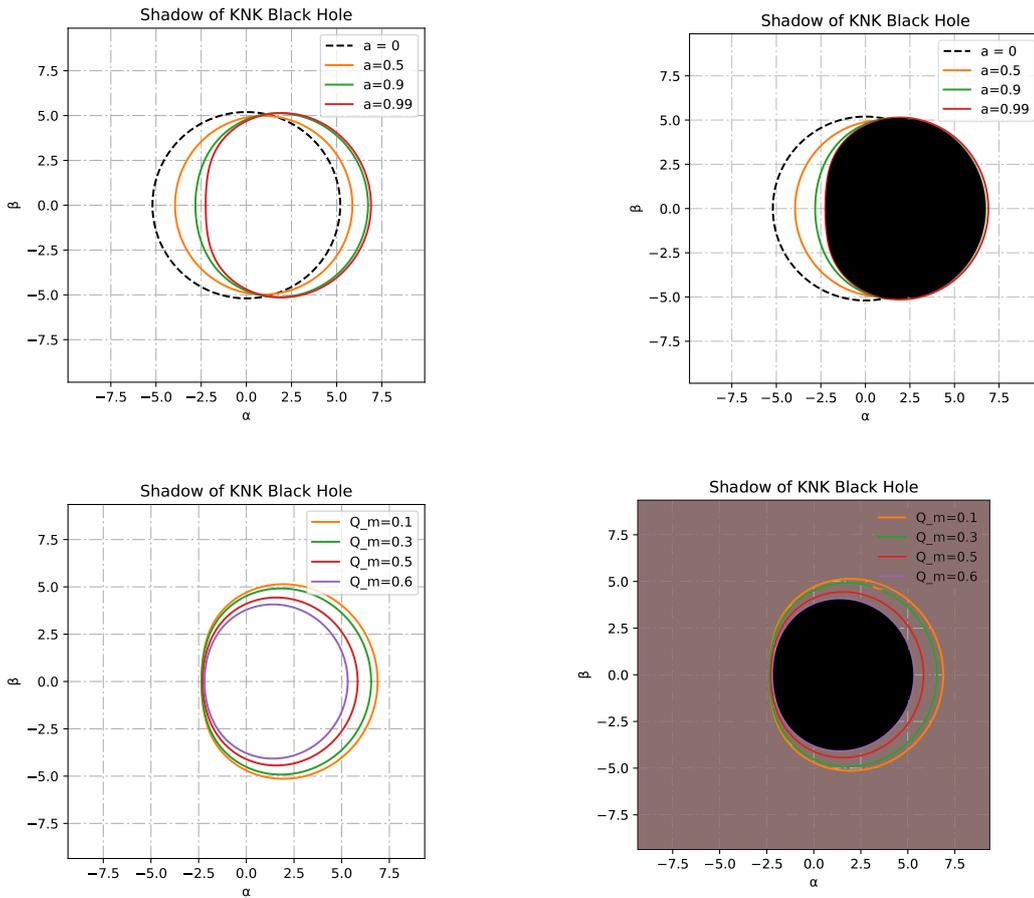


Figure 2. Shadows of KNK black hole with different values of spin a and magnetic charge Q_m . The region bounded by each curve corresponds to the black hole's shadow where the observer is at spatial infinity and in the equatorial plane ($i = \pi/2$). Left/Right side of the figures are prograde circular photon orbit and retrograde circular photon orbit, respectively. Top figures: $Q_e = 0.1$, $Q_m = 0.1$ and $M = 1$. Bottom figures: $Q_e = 0.1$, $M = 1$, and $a = 0.9$.

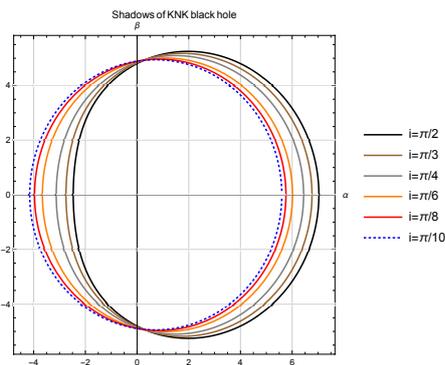


Figure 3. Photon rings are shown at inclination angles i for $Q_e = 0.1$, $Q_m = 0.1$, $M = 1$, and $a = 0.97$.

5 Conclusion

Alongside the recent successes of the LIGO detector and gravitational wave astronomy [111], the EHT has a potential outcome to answer many questions at strong gravitational field regime in general relativity by using the millimeter wavelength radio astronomy [112] and observe the BHS which is embedded on the image of hot gas.

First, we have studied the weak gravitational lensing by using the GBT for the KNK optical spacetime, where the optical geometry gives a more geometrical view on weak gravitational lensing than the quasi-Newtonian lensing method. Remarkably, the method of GBT is easier than the null geodesics method since we only use the optical geometry dealing with spatial light rays. As a consequence of the GBT, the result obtained has a global effect. When the locations of receiver and source are at null infinity, the deflection angle in the weak field approximation has been found as follows eq. (3.15):

$$\hat{\alpha} = \frac{4M}{b} - \frac{3Q_e^2}{4b^2} - \frac{3Q_m^2}{4b^2} \pm \frac{4aM}{b^2}, \quad (5.1)$$

which has the Kerr limit at $Q_e = Q_m = 0$ and the non-rotating dyon black hole when $a = 0$. Furthermore, the deflection angle of the charged black hole is recovered if $a = Q_m = 0$ and the deflection angle of the Schwarzschild black hole is found for $a = Q_e = Q_m = 0$. It is worth noting that the deflection angle is linearly decreased with the extra magnetic charge compared to Kerr-Newman black hole.

Secondly, we have investigated the shadow cast of the KNK black hole. With the choice of this black hole, we have analyzed how a , Q_e , and Q_m parameters affect the image of the shadow. From the numerical plots, we have shown that the magnetic charge or magnetic monopole Q_m dramatically decreases the size of the BHS cast. On the other hand, as seen in figure 3, the shadow cast of KNK black hole decreases with decreasing inclination angle i .

In the near future, we are very hopeful that the EHT will provide the event horizon visualization within black hole shadow. If this observation occurs, we expect to see some of the experimental results of this theoretical work.

Acknowledgments

This work is supported by Comisión Nacional de Ciencias y Tecnología of Chile (CONICYT) through FONDECYT Grant No 3170035 (A. Ö.). A. Ö. is grateful to Institute for Advanced Study, Princeton for hospitality.

References

- [1] J.P. Luminet, *Image of a spherical black hole with thin accretion disk*, *Astron. Astrophys.* **75** (1979) 228.
- [2] P.V.P. Cunha, C.A.R. Herdeiro, E. Radu and H.F. Runarsson, *Shadows of Kerr black holes with scalar hair*, *Phys. Rev. Lett.* **115** (2015) 211102 [[arXiv:1509.00021](#)] [[INSPIRE](#)].
- [3] P.V.P. Cunha and C.A.R. Herdeiro, *Shadows and strong gravitational lensing: a brief review*, *Gen. Rel. Grav.* **50** (2018) 42 [[arXiv:1801.00860](#)] [[INSPIRE](#)].
- [4] H. Falcke, F. Melia and E. Agol, *Viewing the shadow of the black hole at the galactic center*, *Astrophys. J.* **528** (2000) L13 [[astro-ph/9912263](#)] [[INSPIRE](#)].

- [5] G.R. Tremblay et al., *Cold, clumpy accretion onto an active supermassive black hole*, *Nature* **534** (2016) 218 [[arXiv:1606.02304](#)] [[INSPIRE](#)].
- [6] Z.-Q. Shen, K.Y. Lo, M.-C. Liang, P.T.P. Ho and J.-H. Zhao, *A size of ~ 1 au for the radio source Sgr A* at the centre of the Milky Way*, *Nature* **438** (2005) 62 [[astro-ph/0512515](#)] [[INSPIRE](#)].
- [7] L. Huang, M. Cai, Z.-Q. Shen and F. Yuan, *Black hole shadow image and visibility analysis of Sagittarius A**, *Mon. Not. Roy. Astron. Soc.* **379** (2007) 833 [[astro-ph/0703254](#)] [[INSPIRE](#)].
- [8] T. Johannsen, *Sgr A* and general relativity*, *Class. Quant. Grav.* **33** (2016) 113001 [[arXiv:1512.03818](#)] [[INSPIRE](#)].
- [9] P.V.P. Cunha, C.A.R. Herdeiro and M.J. Rodriguez, *Does the black hole shadow probe the event horizon geometry?*, *Phys. Rev. D* **97** (2018) 084020 [[arXiv:1802.02675](#)] [[INSPIRE](#)].
- [10] J.L. Synge, *The escape of photons from gravitationally intense stars*, *Mon. Not. Roy. Astron. Soc.* **131** (1966) 463.
- [11] J.M. Bardeen, *Timelike and null geodesics in the Kerr metric*, in *Black holes (les astres occlus)*, C. Dewitt and B.S. Dewitt eds., Gordon and Breach, New York, U.S.A., (1973), pg. 215 [[INSPIRE](#)].
- [12] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford University Press, New York, U.S.A., (1998).
- [13] K. Hioki and K.-I. Maeda, *Measurement of the Kerr spin parameter by observation of a compact object's shadow*, *Phys. Rev. D* **80** (2009) 024042 [[arXiv:0904.3575](#)] [[INSPIRE](#)].
- [14] T. Johannsen and D. Psaltis, *Testing the no-hair theorem with observations in the electromagnetic spectrum: II. Black-hole images*, *Astrophys. J.* **718** (2010) 446 [[arXiv:1005.1931](#)] [[INSPIRE](#)].
- [15] P.G. Nedkova, V.K. Tinchev and S.S. Yazadjiev, *Shadow of a rotating traversable wormhole*, *Phys. Rev. D* **88** (2013) 124019 [[arXiv:1307.7647](#)] [[INSPIRE](#)].
- [16] L. Amarilla and E.F. Eiroa, *Shadow of a Kaluza-Klein rotating dilaton black hole*, *Phys. Rev. D* **87** (2013) 044057 [[arXiv:1301.0532](#)] [[INSPIRE](#)].
- [17] A. Abdujabbarov, F. Atamurotov, Y. Kucukakca, B. Ahmedov and U. Camci, *Shadow of Kerr-Taub-NUT black hole*, *Astrophys. Space Sci.* **344** (2013) 429 [[arXiv:1212.4949](#)] [[INSPIRE](#)].
- [18] A. Grenzebach, V. Perlick and C. Lämmerzahl, *Photon regions and shadows of Kerr-Newman-NUT black holes with a cosmological constant*, *Phys. Rev. D* **89** (2014) 124004 [[arXiv:1403.5234](#)] [[INSPIRE](#)].
- [19] T. Johannsen et al., *Testing general relativity with the shadow size of Sgr A**, *Phys. Rev. Lett.* **116** (2016) 031101 [[arXiv:1512.02640](#)] [[INSPIRE](#)].
- [20] S.B. Giddings, *Possible observational windows for quantum effects from black holes*, *Phys. Rev. D* **90** (2014) 124033 [[arXiv:1406.7001](#)] [[INSPIRE](#)].
- [21] F. Atamurotov, A. Abdujabbarov and B. Ahmedov, *Shadow of rotating non-Kerr black hole*, *Phys. Rev. D* **88** (2013) 064004 [[INSPIRE](#)].
- [22] S.-W. Wei and Y.-X. Liu, *Observing the shadow of Einstein-Maxwell-dilaton-axion black hole*, *JCAP* **11** (2013) 063 [[arXiv:1311.4251](#)] [[INSPIRE](#)].
- [23] N. Sakai, H. Saida and T. Tamaki, *Gravastar shadows*, *Phys. Rev. D* **90** (2014) 104013 [[arXiv:1408.6929](#)] [[INSPIRE](#)].
- [24] V. Perlick, O. Yu. Tsupko and G.S. Bisnovatyi-Kogan, *Influence of a plasma on the shadow of a spherically symmetric black hole*, *Phys. Rev. D* **92** (2015) 104031 [[arXiv:1507.04217](#)] [[INSPIRE](#)].

- [25] A.A. Abdujabbarov, L. Rezzolla and B.J. Ahmedov, *A coordinate-independent characterization of a black hole shadow*, *Mon. Not. Roy. Astron. Soc.* **454** (2015) 2423 [[arXiv:1503.09054](#)] [[INSPIRE](#)].
- [26] V.K. Tinchev and S.S. Yazadjiev, *Possible imprints of cosmic strings in the shadows of galactic black holes*, *Int. J. Mod. Phys. D* **23** (2014) 1450060 [[arXiv:1311.1353](#)] [[INSPIRE](#)].
- [27] M. Wang, S. Chen and J. Jing, *Chaotic shadow of a non-Kerr rotating compact object with quadrupole mass moment*, [arXiv:1801.02118](#) [[INSPIRE](#)].
- [28] L. Amarilla, E.F. Eiroa and G. Giribet, *Null geodesics and shadow of a rotating black hole in extended Chern-Simons modified gravity*, *Phys. Rev. D* **81** (2010) 124045 [[arXiv:1005.0607](#)] [[INSPIRE](#)].
- [29] A. Yumoto, D. Nitta, T. Chiba and N. Sugiyama, *Shadows of multi-black holes: analytic exploration*, *Phys. Rev. D* **86** (2012) 103001 [[arXiv:1208.0635](#)] [[INSPIRE](#)].
- [30] R. Takahashi, *Black hole shadows of charged spinning black holes*, *Publ. Astron. Soc. Jap.* **57** (2005) 273 [[astro-ph/0505316](#)] [[INSPIRE](#)].
- [31] U. Papnoi, F. Atamurotov, S.G. Ghosh and B. Ahmedov, *Shadow of five-dimensional rotating Myers-Perry black hole*, *Phys. Rev. D* **90** (2014) 024073 [[arXiv:1407.0834](#)] [[INSPIRE](#)].
- [32] J. Dexter and P.C. Fragile, *Tilted black hole accretion disc models of Sagittarius A*: time-variable millimetre to near-infrared emission*, *Mon. Not. Roy. Astron. Soc.* **432** (2013) 2252 [[arXiv:1204.4454](#)] [[INSPIRE](#)].
- [33] J.W. Moffat, *Modified gravity black holes and their observable shadows*, *Eur. Phys. J. C* **75** (2015) 130 [[arXiv:1502.01677](#)] [[INSPIRE](#)].
- [34] Z. Younsi, A. Zhidenko, L. Rezzolla, R. Konoplya and Y. Mizuno, *New method for shadow calculations: application to parametrized axisymmetric black holes*, *Phys. Rev. D* **94** (2016) 084025 [[arXiv:1607.05767](#)] [[INSPIRE](#)].
- [35] T. Johannsen, *Photon rings around Kerr and Kerr-like black holes*, *Astrophys. J.* **777** (2013) 170 [[arXiv:1501.02814](#)] [[INSPIRE](#)].
- [36] A.F. Zakharov, *Constraints on a charge in the Reissner-Nordström metric for the black hole at the galactic center*, *Phys. Rev. D* **90** (2014) 062007 [[arXiv:1407.7457](#)] [[INSPIRE](#)].
- [37] P.V.P. Cunha, J. Grover, C. Herdeiro, E. Radu, H. Runarsson and A. Wittig, *Chaotic lensing around boson stars and Kerr black holes with scalar hair*, *Phys. Rev. D* **94** (2016) 104023 [[arXiv:1609.01340](#)] [[INSPIRE](#)].
- [38] B. Freivogel, R. Jefferson, L. Kabir, B. Mosk and I.-S. Yang, *Casting shadows on holographic reconstruction*, *Phys. Rev. D* **91** (2015) 086013 [[arXiv:1412.5175](#)] [[INSPIRE](#)].
- [39] P.V.P. Cunha, C.A.R. Herdeiro, E. Radu and H.F. Runarsson, *Shadows of Kerr black holes with and without scalar hair*, *Int. J. Mod. Phys. D* **25** (2016) 1641021 [[arXiv:1605.08293](#)] [[INSPIRE](#)].
- [40] T. Ohgami and N. Sakai, *Wormhole shadows*, *Phys. Rev. D* **91** (2015) 124020 [[arXiv:1704.07065](#)] [[INSPIRE](#)].
- [41] A.F. Zakharov, F. De Paolis, G. Ingrosso and A.A. Nucita, *Shadows as a tool to evaluate black hole parameters and a dimension of spacetime*, *New Astron. Rev.* **56** (2012) 64 [[INSPIRE](#)].
- [42] R.A. Hennigar, M.B.J. Poshteh and R.B. Mann, *Shadows, signals and stability in Einsteinian cubic gravity*, *Phys. Rev. D* **97** (2018) 064041 [[arXiv:1801.03223](#)] [[INSPIRE](#)].
- [43] H.-Y. Pu, K. Akiyama and K. Asada, *The effects of accretion flow dynamics on the black hole shadow of Sagittarius A**, *Astrophys. J.* **831** (2016) 4 [[arXiv:1608.03035](#)] [[INSPIRE](#)].
- [44] M. Sharif and S. Iftikhar, *Shadow of a charged rotating non-commutative black hole*, *Eur. Phys. J. C* **76** (2016) 630 [[arXiv:1611.00611](#)] [[INSPIRE](#)].

- [45] A. Abdujabbarov, M. Amir, B. Ahmedov and S.G. Ghosh, *Shadow of rotating regular black holes*, *Phys. Rev. D* **93** (2016) 104004 [[arXiv:1604.03809](#)] [[INSPIRE](#)].
- [46] Z. Xu, X. Hou and J. Wang, *Possibility of identifying matter around rotating black hole with black hole shadow*, [arXiv:1806.09415](#) [[INSPIRE](#)].
- [47] G. Gyulchev, P. Nedkova, V. Tinchev and S. Yazadjiev, *On the shadow of rotating traversable wormholes*, *Eur. Phys. J. C* **78** (2018) 544 [[arXiv:1805.11591](#)] [[INSPIRE](#)].
- [48] T. Vetsov, G. Gyulchev and S. Yazadjiev, *Shadows of black holes in vector-tensor Galileons modified gravity*, [arXiv:1801.04592](#) [[INSPIRE](#)].
- [49] X. Hou, Z. Xu, M. Zhou and J. Wang, *Black hole shadow of Sgr A* in dark matter halo*, *JCAP* **07** (2018) 015 [[arXiv:1804.08110](#)] [[INSPIRE](#)].
- [50] V.I. Dokuchaev and N.O. Nazarova, *Event horizon image within black hole shadow*, [arXiv:1804.08030](#) [[INSPIRE](#)].
- [51] Y. Mizuno et al., *The current ability to test theories of gravity with black hole shadows*, *Nat. Astron.* **2** (2018) 585 [[arXiv:1804.05812](#)] [[INSPIRE](#)].
- [52] V. Perlick, O. Yu. Tsupko and G.S. Bisnovatyi-Kogan, *Black hole shadow in an expanding universe with a cosmological constant*, *Phys. Rev. D* **97** (2018) 104062 [[arXiv:1804.04898](#)] [[INSPIRE](#)].
- [53] Z. Stuchlík, D. Charbulák and J. Schee, *Light escape cones in local reference frames of Kerr-de Sitter black hole spacetimes and related black hole shadows*, *Eur. Phys. J. C* **78** (2018) 180 [[INSPIRE](#)].
- [54] R. Shaikh, P. Kocherlakota, R. Narayan and P.S. Joshi, *Shadows of spherically symmetric black holes and naked singularities*, [arXiv:1802.08060](#) [[INSPIRE](#)].
- [55] E.F. Eiroa and C.M. Sendra, *Shadow cast by rotating braneworld black holes with a cosmological constant*, *Eur. Phys. J. C* **78** (2018) 91 [[arXiv:1711.08380](#)] [[INSPIRE](#)].
- [56] M. Mars, C.F. Paganini and M.A. Oancea, *The fingerprints of black holes-shadows and their degeneracies*, *Class. Quant. Grav.* **35** (2018) 025005 [[arXiv:1710.02402](#)] [[INSPIRE](#)].
- [57] M. Wang, S. Chen and J. Jing, *Shadow casted by a Konoplya-Zhidenko rotating non-Kerr black hole*, *JCAP* **10** (2017) 051 [[arXiv:1707.09451](#)] [[INSPIRE](#)].
- [58] N. Tsukamoto, *Black hole shadow in an asymptotically-flat, stationary and axisymmetric spacetime: the Kerr-Newman and rotating regular black holes*, *Phys. Rev. D* **97** (2018) 064021 [[arXiv:1708.07427](#)] [[INSPIRE](#)].
- [59] B.P. Singh and S.G. Ghosh, *Shadow of Schwarzschild-Tangherlini black holes*, *Annals Phys.* **395** (2018) 127 [[arXiv:1707.07125](#)] [[INSPIRE](#)].
- [60] J.R. Mureika and G.U. Varieschi, *Black hole shadows in fourth-order conformal Weyl gravity*, *Can. J. Phys.* **95** (2017) 1299 [[arXiv:1611.00399](#)] [[INSPIRE](#)].
- [61] Y. Huang, S. Chen and J. Jing, *Double shadow of a regular phantom black hole as photons couple to the Weyl tensor*, *Eur. Phys. J. C* **76** (2016) 594 [[arXiv:1606.04634](#)] [[INSPIRE](#)].
- [62] M. Ghasemi-Nodehi and C. Bambi, *Note on a new parametrization for testing the Kerr metric*, *Eur. Phys. J. C* **76** (2016) 290 [[arXiv:1604.07032](#)] [[INSPIRE](#)].
- [63] N. Tsukamoto, Z. Li and C. Bambi, *Constraining the spin and the deformation parameters from the black hole shadow*, *JCAP* **06** (2014) 043 [[arXiv:1403.0371](#)] [[INSPIRE](#)].
- [64] F.H. Vincent, E. Gourgoulhon, C. Herdeiro and E. Radu, *Astrophysical imaging of Kerr black holes with scalar hair*, *Phys. Rev. D* **94** (2016) 084045 [[arXiv:1606.04246](#)] [[INSPIRE](#)].
- [65] M. Bartelmann and P. Schneider, *Weak gravitational lensing*, *Phys. Rept.* **340** (2001) 291 [[astro-ph/9912508](#)] [[INSPIRE](#)].

- [66] V. Bozza, *Gravitational lensing by black holes*, *Gen. Rel. Grav.* **42** (2010) 2269 [[arXiv:0911.2187](#)] [[INSPIRE](#)].
- [67] K.S. Virbhadra and C.R. Keeton, *Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities*, *Phys. Rev. D* **77** (2008) 124014 [[arXiv:0710.2333](#)] [[INSPIRE](#)].
- [68] S. Zschocke, *A generalized lens equation for light deflection in weak gravitational fields*, *Class. Quant. Grav.* **28** (2011) 125016 [[arXiv:1105.3621](#)] [[INSPIRE](#)].
- [69] I.Z. Stefanov, S.S. Yazadjiev and G.G. Gylulchev, *Connection between black-hole quasinormal modes and lensing in the strong deflection limit*, *Phys. Rev. Lett.* **104** (2010) 251103 [[arXiv:1003.1609](#)] [[INSPIRE](#)].
- [70] T. Manna, F. Rahaman, S. Molla, J. Bhadra and H.H. Shah, *Strong lensing of a regular black hole with an electrodynamics source*, *Gen. Rel. Grav.* **50** (2018) 54 [[arXiv:1802.02334](#)] [[INSPIRE](#)].
- [71] T. Ono, A. Ishihara and H. Asada, *Deflection angle of light for an observer and source at finite distance from a rotating wormhole*, *Phys. Rev. D* **98** (2018) 044047 [[arXiv:1806.05360](#)] [[INSPIRE](#)].
- [72] W. Hu, *Weak lensing of the CMB: a harmonic approach*, *Phys. Rev. D* **62** (2000) 043007 [[astro-ph/0001303](#)] [[INSPIRE](#)].
- [73] G.W. Gibbons and M.C. Werner, *Applications of the Gauss-Bonnet theorem to gravitational lensing*, *Class. Quant. Grav.* **25** (2008) 235009 [[arXiv:0807.0854](#)] [[INSPIRE](#)].
- [74] G.W. Gibbons and C.M. Warnick, *Universal properties of the near-horizon optical geometry*, *Phys. Rev. D* **79** (2009) 064031 [[arXiv:0809.1571](#)] [[INSPIRE](#)].
- [75] M.C. Werner, *Gravitational lensing in the Kerr-Randers optical geometry*, *Gen. Rel. Grav.* **44** (2012) 3047 [[arXiv:1205.3876](#)] [[INSPIRE](#)].
- [76] G.W. Gibbons, *The Jacobi-metric for timelike geodesics in static spacetimes*, *Class. Quant. Grav.* **33** (2016) 025004 [[arXiv:1508.06755](#)] [[INSPIRE](#)].
- [77] G.W. Gibbons and M. Vyska, *The application of Weierstrass elliptic functions to Schwarzschild null geodesics*, *Class. Quant. Grav.* **29** (2012) 065016 [[arXiv:1110.6508](#)] [[INSPIRE](#)].
- [78] G.W. Gibbons, C.A.R. Herdeiro, C.M. Warnick and M.C. Werner, *Stationary metrics and optical Zermelo-Randers-Finsler geometry*, *Phys. Rev. D* **79** (2009) 044022 [[arXiv:0811.2877](#)] [[INSPIRE](#)].
- [79] I. Sakalli and A. Övgün, *Hawking radiation and deflection of light from Rindler modified Schwarzschild black hole*, *EPL* **118** (2017) 60006 [[arXiv:1702.04636](#)] [[INSPIRE](#)].
- [80] K. Jusufi, M.C. Werner, A. Banerjee and A. Övgün, *Light deflection by a rotating global monopole spacetime*, *Phys. Rev. D* **95** (2017) 104012 [[arXiv:1702.05600](#)] [[INSPIRE](#)].
- [81] G. Crisnejo and E. Gallo, *Weak lensing in a plasma medium and gravitational deflection of massive particles using the Gauss-Bonnet theorem. A unified treatment*, *Phys. Rev. D* **97** (2018) 124016 [[arXiv:1804.05473](#)] [[INSPIRE](#)].
- [82] K. Jusufi, I. Sakalli and A. Övgün, *Effect of Lorentz symmetry breaking on the deflection of light in a cosmic string spacetime*, *Phys. Rev. D* **96** (2017) 024040 [[arXiv:1705.06197](#)] [[INSPIRE](#)].
- [83] A. Övgün, K. Jusufi and I. Sakalli, *Gravitational lensing under the effect of Weyl and bumblebee gravities: applications of Gauss-Bonnet theorem*, [arXiv:1805.09431](#) [[INSPIRE](#)].

- [84] K. Jusufi, A. Övgün and A. Banerjee, *Light deflection by charged wormholes in Einstein-Maxwell-dilaton theory*, *Phys. Rev. D* **96** (2017) 084036 [Addendum *ibid.* **D 96** (2017) 089904] [[arXiv:1707.01416](#)] [[INSPIRE](#)].
- [85] A. Övgün, G. Gylchev and K. Jusufi, *Weak gravitational lensing by phantom black holes and phantom wormholes using the Gauss-Bonnet theorem*, [arXiv:1806.03719](#) [[INSPIRE](#)].
- [86] K. Jusufi and A. Övgün, *Gravitational Lensing by rotating wormholes*, *Phys. Rev. D* **97** (2018) 024042 [[arXiv:1708.06725](#)] [[INSPIRE](#)].
- [87] A. Övgün, *Maxwell's fisheye lensing effect by black holes and Gauss-Bonnet theorem*, [arXiv:1806.05549](#) [[INSPIRE](#)].
- [88] K. Jusufi, F. Rahaman and A. Banerjee, *Semiclassical gravitational effects on the gravitational lensing in the spacetime of topological defects*, *Annals Phys.* **389** (2018) 219 [[arXiv:1709.00227](#)] [[INSPIRE](#)].
- [89] M. Cvetič, G.W. Gibbons and C.N. Pope, *Photon spheres and sonic horizons in black holes from supergravity and other theories*, *Phys. Rev. D* **94** (2016) 106005 [[arXiv:1608.02202](#)] [[INSPIRE](#)].
- [90] K. Jusufi and A. Övgün, *Effect of the cosmological constant on the deflection angle by a rotating cosmic string*, *Phys. Rev. D* **97** (2018) 064030 [[arXiv:1712.01771](#)] [[INSPIRE](#)].
- [91] A. Övgün, *Light deflection by Damour-Solodukhin wormholes and Gauss-Bonnet theorem*, *Phys. Rev. D* **98** (2018) 044033 [[arXiv:1805.06296](#)] [[INSPIRE](#)].
- [92] K. Jusufi, A. Övgün, J. Saavedra, Y. Vásquez and P. González, *Deflection of light by rotating regular black holes using the Gauss-Bonnet theorem*, *Phys. Rev. D* **97** (2018) 124024 [[arXiv:1804.00643](#)] [[INSPIRE](#)].
- [93] A. Ishihara, Y. Suzuki, T. Ono, T. Kitamura and H. Asada, *Gravitational bending angle of light for finite distance and the Gauss-Bonnet theorem*, *Phys. Rev. D* **94** (2016) 084015 [[arXiv:1604.08308](#)] [[INSPIRE](#)].
- [94] A. Ishihara, Y. Suzuki, T. Ono and H. Asada, *Finite-distance corrections to the gravitational bending angle of light in the strong deflection limit*, *Phys. Rev. D* **95** (2017) 044017 [[arXiv:1612.04044](#)] [[INSPIRE](#)].
- [95] H. Arakida, *Light deflection and Gauss-Bonnet theorem: definition of total deflection angle and its applications*, *Gen. Rel. Grav.* **50** (2018) 48 [[arXiv:1708.04011](#)] [[INSPIRE](#)].
- [96] S. Doeleman et al., *Event-horizon-scale structure in the supermassive black hole candidate at the galactic centre*, *Nature* **455** (2008) 78 [[arXiv:0809.2442](#)] [[INSPIRE](#)].
- [97] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, *Nucl. Phys. B* **79** (1974) 276 [[INSPIRE](#)].
- [98] A.M. Polyakov, *Compact gauge fields and the infrared catastrophe*, *Phys. Lett. B* **59** (1975) 82 [[INSPIRE](#)].
- [99] A.M. Polyakov, *Quark confinement and topology of gauge groups*, *Nucl. Phys. B* **120** (1977) 429 [[INSPIRE](#)].
- [100] C. Chakraborty and S. Bhattacharyya, *Does the gravitomagnetic monopole exist? A clue from a black hole X-ray binary*, *Phys. Rev. D* **98** (2018) 043021 [[arXiv:1712.01156](#)] [[INSPIRE](#)].
- [101] M. Kasuya, *Exact solution of a rotating dyon black hole*, *Phys. Rev. D* **25** (1982) 995 [[INSPIRE](#)].
- [102] B.C. Chanyal, *A new development in quantum field equations of dyons*, [arXiv:1712.08512](#) [[INSPIRE](#)].

- [103] MoEDAL collaboration, N.E. Mavromatos and V.A. Mitsou, *Physics reach of MoEDAL at LHC: magnetic monopoles, supersymmetry and beyond*, *EPJ Web Conf.* **164** (2017) 04001 [[arXiv:1612.07012](#)] [[INSPIRE](#)].
- [104] M. Hossain Ali, *Nongeneric SUSY in hot NUT-Kerr-Newman-Kasuya space-time*, *Phys. Lett. B* **549** (2002) 344 [[INSPIRE](#)].
- [105] A. Comtet, *Magnetic monopoles in curved space-times*, *Ann. Inst. H. Poincaré Phys. Theor.* **32** (1980) 283 [[INSPIRE](#)].
- [106] M. Sharif and W. Javed, *Fermions tunneling from Plebanski-Demianski black holes*, *Gen. Rel. Grav.* **45** (2013) 1051 [[arXiv:1309.6384](#)] [[INSPIRE](#)].
- [107] R.M. Wald, *General relativity*, University of Chicago Press, Chicago, U.S.A. and London, U.K., (1984).
- [108] T. Ono, A. Ishihara and H. Asada, *Gravitomagnetic bending angle of light with finite-distance corrections in stationary axisymmetric spacetimes*, *Phys. Rev. D* **96** (2017) 104037 [[arXiv:1704.05615](#)] [[INSPIRE](#)].
- [109] R.P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (1963) 237 [[INSPIRE](#)].
- [110] G. Clement, D. Gal'tsov and C. Leygnac, *Linear dilaton black holes*, *Phys. Rev. D* **67** (2003) 024012 [[hep-th/0208225](#)] [[INSPIRE](#)].
- [111] VIRGO and LIGO SCIENTIFIC collaborations, B.P. Abbott et al., *Observation of gravitational waves from a binary black hole merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [[arXiv:1602.03837](#)] [[INSPIRE](#)].
- [112] *Event Horizon Telescope webpage*, <https://eventhorizontelescope.org>.
- [113] S. Gillessen et al., *Monitoring stellar orbits around the massive black hole in the galactic center*, *Astrophys. J.* **692** (2009) 1075 [[arXiv:0810.4674](#)] [[INSPIRE](#)].