


Quantum tunneling and quasinormal modes in the spacetime of the Alcubierre warp drive

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Received: 12 September 2017 / Accepted: 4 December 2017
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Abstract In a seminal paper, Alcubierre showed that Einstein's theory of general relativity appears to allow a super-luminal motion. In the present study, we use a recent eternal-warp-drive solution found by Alcubierre to study the effect of Hawking radiation upon an observer located within the warp drive in the framework of the quantum tunneling method. We find the same expression for the Hawking temperatures associated with the tunneling of both massive vector and scalar particles, and show this expression to be proportional to the velocity of the warp drive. On the other hand, since the discovery of gravitational waves, the quasinormal modes (QNMs) of black holes have also been extensively studied. With this purpose in mind, we perform a QNM analysis of massive scalar field perturbations in the background of the eternal-Alcubierre-warp-drive spacetime. Our analytical analysis shows that massive scalar perturbations lead to stable QNMs.

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Keywords Warp drive · Super-luminal motion · Alcubierre · Hawking radiation · Quasinormal modes · Scalar · Massive vector particles

1 Introduction

General relativity is a playground for many unexpected solutions. One of them was introduced by M. Alcubierre in 1994, and is known as the eternal-Alcubierre-warp-drive (EAWD) spacetime [1,2]. This solution is found in the original theory of the general relativity and it allows super-luminal motion by expanding (contracting) the spacetime behind (in front of) an observer within the spacetime [3]. It is noted that the observer sits in a locally flat region of the spacetime, which is the so-called warp bubble. However, the energy-momentum tensor of EAWD spacetime violates the energy conditions (weak, dominant and strong) in that weak energy condition requires a negative energy density [4]. Today, it is known that negative energy only arises in certain special cases of QFT, such as the Casimir effect or dark energy in cosmology. On the other hand, the basic premise of EAWD spacetime is widely accepted to have occurred during in the inflationary era of the early universe; when the relative speeds of two co-moving observers in this era are considered, super-luminal motion appears to occur without violating special or general relativity.

Hiscock was the first to study quantum effects in the EAWD spacetime and showed that the stress–energy diverges if the apparent velocity of the warp bubble (around the spacecraft) exceeds the speed of light [5]. Finazzi et al. [6] extended Hiscock’s results by investigating semiclassical instability in dynamical warp drives. Recently, various studies [7–16] have appeared in the literature, inspiring us to use the EAWD metric in our present study.

The discovery of black hole radiation by Hawking and Bekenstein [17–24] showed that black holes are in fact not black, instead they are “gray” and thermally radiate. Today, Hawking radiation is studied in an elegant and widely used way through a quantum tunneling model, among others (see for example [25–69] and references therein). In the present work, we consider the Hawking radiation of massive scalar and vector particles from the EAWD spacetime. The effect of GUP [70] on the Hawking radiation of the EAWD spacetime will also be considered.

QNMs are another important means of probing spacetimes. They represent the characteristic resonance spectrum of a black hole. By taking the back reaction effects into account, it was shown by Parikh and Wilczek [71] that black holes can radiate energy with a non-thermal spectrum. Although QNMs dominate in the response of a black hole to external perturbations, they are affected by Hawking radiation [72–77]. In the last decade, studies of both QNMs and Hawking radiation have gained momentum (see for instance [78–84]). In fact, such works show that the black holes are good testing grounds for quantum gravity theory [85–101]. Along the same line of thought, in the present study we want to explore the quantum gravitational outcomes of the EAWD spacetime by considering QNMs in addition to Hawking radiation.

The Laser-Interferometer-Gravitational-Wave-Observatory (LIGO) has recently made the ‘discovery of the century’ by detecting the gravitational waves originating

from a merger between two black holes [102]. This event once more proves Einstein's theory of the gravity. QNMs of gravitational waves provide critical information about the structure of black holes. Thus, QNMs can be a tool to test general relativity and possible deviations from it [103, 104]. Moreover, in the AdS/CFT correspondence, QNMs are used to study the rapidity of a thermal state at the boundary where thermal equilibrium is established [105]. Today, there are many analytical and numerical works on QNMs in the literature (see for instance [106–138]). To study analytical QNMs, one should solve the field equation on the considered geometry and derive the one dimensional Schrödinger equation or the so-called Zerilli equation [139] in terms of the tortoise coordinate r_* :

$$\frac{d^2\psi}{dr_*^2} = \left[\omega^2 - V(r_*) \right] \psi, \quad (1)$$

where ω is the frequency of the QNM. In general, ω is labeled by a discrete quantum number ($n = 0, 1, 2, \dots$) and has the following asymptotic form [140, 141]

$$\omega_n = (\text{offset}) + in (\text{gap}) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \rightarrow \infty. \quad (2)$$

Here, the “gap” and “offset” are complex parameters that are determined by the precise form of the spacetime-dependent potential barrier seen in Eq. (1). In fact, the real part of ω shows the temporal oscillation and its imaginary part describes the exponential decay.

To compute the QNMs of the EAWD spacetime, we consider the massive Klein–Gordon equation (KGE). We show that the radial part of the massive KGE reduces to a hypergeometric differential equation after some manipulation. Thus, we demonstrate how one can analytically derive the complex QNMs by applying the appropriate boundary conditions.

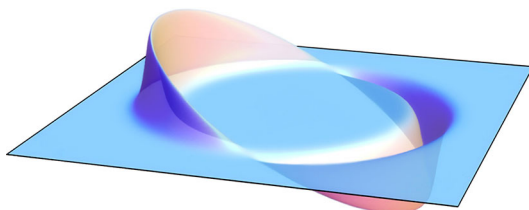
The outline of the paper is as follows. In Sect. 2, we briefly introduce the EAWD spacetime. Section 3 is devoted to calculating the Hawking temperature of this spacetime using the vector particles' quantum tunneling with the help of the semi-classical Hamilton–Jacobi method. In Sect. 6, we consider the quantum tunneling of scalar particles in the EAWD spacetime. We also study the GUP effect of quantum gravity on the EAWD's Hawking radiation. In Sect. 5, we analytically study the massive scalar field perturbations in the background of the EAWD spacetime and represent exact QNMs. We draw our conclusions in Sect. 6.

2 EAWD spacetime

The EAWD spacetime is described by the following line-element, which was derived by Alcubierre [1, 2]:

$$ds^2 = -c^2 dt^2 + [dx - v(r)dt]^2 + dx^2 + dy^2. \quad (3)$$

Fig. 1 Two-dimensional visualization of an Alcubierre drive, showing the opposing regions of expanding and contracting spacetime that displace the central region [142]



Here, $v(r)$ is the velocity of the spacecraft’s moving frame and r represents the distance from the center of the bubble, which is given by

$$r = \sqrt{(x - v_0t)^2 + y^2 + z^2}, \tag{4}$$

where v_0 represents the warp-drive velocity [6]. Furthermore, it is convenient to introduce a new function, $f(r)$, given by $v(r) = v_0f(r)$. $f(r)$ should be a smooth function satisfying the conditions $f(0) = 1$ and $f(r) \rightarrow 0$ if $r \rightarrow \infty$. For the sake of simplicity, in this paper, we shall consider the following 1 + 1 dimensional EAWD metric [6]:

$$ds^2 = -c^2dt^2 + [dx - v(r)dt]^2, \tag{5}$$

where r is given as $r = x - v_0t$. The choice of $f(r)$ is not unique: we select the following simple bell-shaped function [6]

$$f(r) = \frac{1}{\cosh(r/a)}. \tag{6}$$

Note that when $v_0 > c$, we have the super-luminal warp drive. One can write the above metric as a Painleve metric (Fig. 1):

$$ds^2 = -c^2dt^2 + [dr - \bar{v}(r)dt]^2, \tag{7}$$

where

$$\bar{v}(r) = v(r) - v_0. \tag{8}$$

We also note that $\bar{v}(r) < 0$ because the warp drive is right going i.e., $v(r) \leq v_0$, with $v(r) > 0$. Introducing $\alpha = v_0/c$, the shift velocity becomes

$$\bar{v}(r) = \alpha c \left[\frac{1}{\cosh(r/a)} - 1 \right]. \tag{9}$$

It is interesting to point out that two horizons appear when $\alpha > 1$, by setting $\bar{v}(r) = -c$. The center of the bubble is located at $r = 0$, while the horizons are located at

$$r_h = r_{1,2} = \mp a \ln \left(\beta + \sqrt{\beta^2 - 1} \right), \tag{10}$$

with

$$\beta = \frac{\alpha}{\alpha - 1} > 1. \tag{11}$$

3 Quantum tunneling of vector particles in EAWD

The relativistic field equation for a massive vector particles is governed by the Proca equation (PE) [32,33]

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \Psi^{\nu\mu}) + \frac{m^2 c^2}{\hbar^2} \Psi^\nu = 0, \tag{12}$$

where

$$\Psi_{\mu\nu} = \partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu. \tag{13}$$

In our setup, we shall solve the PE in the background of EAWD spacetime (7). The solution is proposed in terms of the WKB approximation as follows

$$\Psi_\nu = C_\nu(t, r) \exp\left(\frac{i}{\hbar} (S_0(t, r) + \hbar S_1(t, r) + \dots)\right). \tag{14}$$

Using Eqs. (12) and (14), one finds

$$0 = \left[\frac{-m^2 c^2 \bar{v}(r) + (\partial_t S_0(t, r))(\partial_r S_0(t, r))}{c^2} \right] C_1 - \left[\frac{(\partial_r S_0)^2 + m^2 c^2}{c^2} \right] C_2, \tag{15}$$

$$0 = \left[\frac{-(\partial_t S_0(t, r))^2 + (c^2 - \bar{v}^2(r))m^2 c^2}{c^2} \right] C_1 + \left[\frac{-m^2 c^2 \bar{v}(r) + (\partial_t S_0(t, r))(\partial_r S_0(t, r))}{c^2} \right] C_2. \tag{16}$$

By using the symmetries of the spacetime (7), one may choose the action as

$$S_0(t, r) = -Et + R(r), \tag{17}$$

then by making a substitution from Eqs. (15) and (16), we find out

$$\left[\frac{-m^2 c^2 \bar{v}(r) - ER'(r)}{c^2} \right] C_1 - \left[\frac{(R'(r))^2 + m^2 c^2}{c^2} \right] C_2 = 0, \tag{18}$$

$$\left[\frac{-E^2 + (c^2 - \bar{v}^2(r))m^2 c^2}{c^2} \right] C_1 + \left[\frac{-m^2 c^2 \bar{v}(r) - ER'(r)}{c^2} \right] C_2 = 0. \tag{19}$$

The physical meaning from these equations can be revealed after we consider a 2×2 matrix equation. In particular, we can choose a matrix Ξ and multiply it with a transpose of a vector (C_1, C_2) , yielding the matrix equation

$$\Xi (C_1, C_2)^T = 0. \tag{20}$$

This matrix has the following non-zero elements

$$\Xi_{11} = \Xi_{22} = \frac{-m^2 c^2 \bar{v}(r) - E R'(r)}{c^2}, \tag{21}$$

$$\Xi_{12} = -\frac{R'^2 + m^2 c^2}{c^2}, \tag{22}$$

$$\Xi_{21} = \frac{-E^2 + (c^2 - \bar{v}^2(r))m^2 c^2}{c^2}. \tag{23}$$

In order to solve for the radial solution we need to consider $\det \Xi = 0$, which leads to the following differential equation

$$-\frac{m^2 ((-c^2 + (\bar{v}(r))^2) R'^2 - 2 E \bar{v}(r) R'(r) - c^4 m^2 + E^2)}{c^2} = 0. \tag{24}$$

This equation can be easily solved for the radial part

$$R_{\mp} = \int \frac{E \bar{v}(r) \mp \sqrt{c^4 m^2 (\bar{v}(r) - c)^2 + E^2 c^2}}{(\bar{v}(r) - c)(\bar{v}(r) + c)} dr. \tag{25}$$

This integral is singular at the horizon $\bar{v}(r_h = r_{1,2}) = -c$. Thus, by expanding in series the velocity near the horizon we find

$$\bar{v}(r_h) = -c + \kappa_{1,2} (r - r_{1,2}) + \mathcal{O} \left[(r - r_{1,2})^2 \right], \tag{26}$$

where we have defined the surface gravity as

$$\kappa_{1,2} = \left. \frac{dv(r)}{dr} \right|_{r_h=r_{1,2}} = \pm \frac{c(\alpha - 1)\sqrt{\beta^2 - 1}}{a\beta} \equiv \pm \kappa. \tag{27}$$

It is not difficult so see that we find non-zero contribution only for the ingoing particles moving from the outside to the inside of the EAWD

$$R_-(r_h = r_{1,2}) = \int \frac{E}{\kappa_{1,2}} \frac{dr}{(r - r_{1,2})}. \tag{28}$$

As was noted in [6], the surface gravity associated with the first horizon is positive $\kappa_1 = \kappa > 0$, while the one associated with the second horizon is negative $\kappa_2 = -\kappa < 0$. The physical significance of these two horizons is that they represent a black and a white horizon, respectively. As was pointed out in Ref. [6], the choose of the horizon does not lead to a different Hawking temperature in absolute value, therefore we may consider the two surface gravities to have the same absolute value i.e. $\kappa = |\kappa_{1,2}|$. It is worth noting that ‘in 3 + 1 EAWD spacetime’, while one side of the spacetime is expanding (white horizon), the other side of the spacetime contracts (black horizon). In other words, when the particles tunnel from outside to inside near

the black horizon, the particles in the vicinity of the white horizon conversely tunnel from inside to outside. The tunneling rate is related to the imaginary part of the action in the classically forbidden region, which is given by

$$\Gamma \sim e^{-2\text{Im}S}. \tag{29}$$

Therefore, we can use the following identity

$$\lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{r - r_h \pm i\epsilon} = \pi \delta(r - r_h), \tag{30}$$

and find a non-zero contribution only for the ingoing radial part $R_-(r)$:

$$\text{Im}R_-(r) = \frac{\pi E}{\kappa}, \quad \text{Im}R_+(r) = 0. \tag{31}$$

We can define the tunneling probability from outside to inside the warp drive by the following relation

$$\Gamma = \frac{\exp(-2\text{Im}R_-)}{\exp(-2\text{Im}R_+)} = \exp\left(-\frac{2\pi E}{\kappa}\right). \tag{32}$$

Comparing the above relation with the Boltzmann equation $\Gamma_B = \exp(-E/T)$, we get the Hawking temperature as follows

$$T_H = \frac{\kappa}{2\pi} = v_0 \frac{f'(r_h)}{2\pi}. \tag{33}$$

Equation (33) shows that an observer inside the warp drive experiences a thermal flux of Hawking quanta. In fact, this phenomenon is different from the black hole radiation in which the Hawking quanta tunnel from inside to outside of the horizon. However, the right way to recover the above result for the Hawking temperature is to consider the invariance of canonical transformations [51,52,55,56,66,67]. Namely, by considering a closed path, which goes from outside, $r = r_i$, (i.e, just outside of the horizon) to $r = r_f$ (just inside of the horizon):

$$\oint p_r dr = \int_{r_i}^{r_f} p_r^{in} dr + \int_{r_f}^{r_i} p_r^{out} dr, \tag{34}$$

the total quantum tunneling rate is found to be a sum of the spatial and temporal contributions. To see this, we draw attention to the pole located at the horizon $r = r_h = r_{1,2}$ seen in Eq. (25) with $\bar{v}(r_h) = -c$. But, we can shift the pole by using the Feynman prescription i.e., $r_h \rightarrow r_h + i\epsilon$:

$$\text{Im} \oint p_r dr = \lim_{\epsilon \rightarrow 0} \left\{ \text{Im} \oint \frac{E\bar{v}(r_h) \mp \sqrt{c^4 m^2 (\bar{v}^2(r_h) - c^2) + E^2 c^2}}{(\bar{v}(r_h) - c)\kappa(r_h)(r - r_h \pm i\epsilon)} dr \right\}, \tag{35}$$

in which the relation $p_r = \partial_r R$ is used. Furthermore, the above equation can also be written as

$$\begin{aligned} \text{Im} \oint p_r dr &= \lim_{\epsilon \rightarrow 0} \left[\text{Im} \int_{r_i}^{r_f} \frac{E\bar{v}(r_h) + \sqrt{c^4 m^2 (\bar{v}^2(r_h) - c^2) + E^2 c^2}}{(\bar{v}(r_h) - c)\kappa(r_h)(r - r_h \pm i\epsilon)} dr \right] \\ &+ \lim_{\epsilon \rightarrow 0} \left[\text{Im} \int_{r_f}^{r_i} \frac{E\bar{v}(r_h) - \sqrt{c^4 m^2 (\bar{v}^2(r_h) - c^2) + E^2 c^2}}{(\bar{v}(r_h) - c)\kappa(r_h)(r - r_h \pm i\epsilon)} dr \right]. \end{aligned} \tag{36}$$

One can see from the last equation that there is no contribution to the imaginary part coming from the first term. This is due to the Painleve coordinates which means that the particle experiences barrier only from outside the horizon to inside (not from the reverse way). Hence, the spatial contribution reads

$$\text{Im} \oint p_r dr = \frac{\pi E}{\kappa}. \tag{37}$$

We shall now proceed to find the temporal contribution. To this end, we express the EAWD spacetime in the following compact form (Painleve coordinates):

$$ds^2 = -\left(c^2 - \bar{v}^2(r)\right)dt^2 + \frac{c^2}{c^2 - \bar{v}^2(r)}dr^2, \tag{38}$$

under the coordinate transformations

$$dt = d\tilde{t} - \frac{\bar{v}(r)}{c^2 - \bar{v}^2(r)}dr, \tag{39}$$

where t is the Painleve time. Using the action Eq. (17), we find

$$S_0 = -E\tilde{t} + \int \frac{\bar{v}(r) E}{c^2 - \bar{v}^2(r)}dr + R(r) \tag{40}$$

By solving the integral, we can find the temporal contribution as

$$\text{Im}(E\Delta t^{out,in}) = \frac{\pi E}{2\kappa}. \tag{41}$$

Following the paper of Akhmedova et al [51,52], the total tunneling rate is obtained as follows

$$\begin{aligned} \Gamma &= \exp \left[-\left(\text{Im}(E\Delta t^{out}) + \text{Im}(E\Delta t^{in}) + \text{Im} \oint p_r dr \right) \right] \\ &= \exp \left(-\frac{2\pi E}{\kappa} \right). \end{aligned} \tag{42}$$

Employing the Boltzmann formula $\Gamma_B = e^{-E/T_H}$, we find the foreknown Hawking temperature:

$$T_H = \frac{\kappa}{2\pi} = v_0 \frac{f'(r_h)}{2\pi}. \tag{43}$$

4 Quantum tunneling of scalar particles and the effect of GUP on Hawking radiation

In this section, we shall focus on the Hawking temperature by using the tunneling of the scalar particle from the EAWD spacetime. We start with the KGE

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - \frac{m^2 c^2}{\hbar^2} \Phi = 0, \tag{44}$$

on the background of the metric (38), which can be rewritten as

$$ds^2 = -F dt^2 + \frac{1}{G} dr^2, \tag{45}$$

where $F = c^2 - \bar{v}^2(r)$ and $G = \frac{c^2 - \bar{v}^2(r)}{c^2}$. From those two equations, we obtain

$$-\frac{1}{F} \partial_t^2 \Phi + G \partial_r^2 \Phi + \frac{1}{2} \frac{G}{F} F' \partial_r \Phi + \frac{1}{2} G' \partial_r \Phi - \frac{m^2 c^2}{\hbar^2} \Phi = 0. \tag{46}$$

Then, we make use of the WKB ansatz:

$$\Phi = \exp\left(\frac{i}{\hbar} S(r, t)\right). \tag{47}$$

In the limit of \hbar goes to zero, we obtain the relativistic Hamilton–Jacobi equation:

$$\frac{1}{F} (\partial_t S)^2 - G (\partial_r S)^2 - m^2 c^2 = 0. \tag{48}$$

We now suppose the solution of the action as the following

$$S(r, t) = -\omega t + W(r). \tag{49}$$

The radial part $W(r)$ is obtained from

$$W(r) = \pm \int \frac{dr}{\sqrt{FG}} \sqrt{E^2 - m^2 c^2 F}. \tag{50}$$

The above integral is nothing but the

$$W(r) = \pm \int \frac{c \sqrt{E^2 - m^2 c^2 F}}{(c - \bar{v}(r))(c + \bar{v}(r))} dr. \tag{51}$$

Note that there is a pole at the horizon $F(r_h) = 0$. After using the near horizon approximation and residue theorem, we solve the complex integral and find the following solution:

$$W(r_h) = \pm \frac{\pi i E}{2 \kappa(r_h)} + \text{real contribution.} \tag{52}$$

The spatial contribution to the tunneling can be calculated as

$$\begin{aligned} \Gamma_{spatial} &\propto \exp\left(-\text{Im} \oint p_r dr\right) \\ &= \exp\left[-\text{Im} \left(\int p_r^+ dr - \int p_r^- dr\right)\right] \\ &= \exp\left(-\frac{\pi E}{\kappa(r_h)}\right). \end{aligned} \tag{53}$$

The temporal part contribution is revealed after we consider the connection of the interior region and the exterior region of the EAWD spacetime. Introducing $t \rightarrow t - i\pi/(2\kappa)$, we will have $\text{Im}(E \Delta t^{out,in}) = E\pi/(2\kappa)$. Then the total temporal contribution for a round trip gives

$$\begin{aligned} \Gamma_{temporal} &\propto \exp\left[-\left(\text{Im}(E \Delta t^{out}) + \text{Im}(E \Delta t^{in})\right)\right] \\ &= \exp\left(-\frac{\pi E}{\kappa(r_h)}\right). \end{aligned} \tag{54}$$

The total tunneling rate of the particles tunneling from outside to the inside reads

$$\Gamma = \exp\left[-\left(\text{Im}(E \Delta t^{out}) + \text{Im}(E \Delta t^{in}) + \text{Im} \oint p_r dr\right)\right] = \exp\left[-\frac{2\pi E}{\kappa(r_h)}\right].$$

Hence the Hawking temperature is obtained as

$$T_H = \frac{\kappa}{2\pi} = v_0 \frac{f'(r_h)}{2\pi}, \tag{55}$$

in full agreement to previous result. Let us now consider the GUP effect on the scalar particle tunneling the EAWD black hole using the modified commutation relations and modification of the KGE with GUP [44–50], Φ as follows:

$$-(i\hbar)^2 \partial^t \partial_t \Phi = \left[(i\hbar)^2 \partial^i \partial_i + m_p^2 c^2 \right] \left[1 - 2\alpha_{GUP} \left((i\hbar)^2 \partial^i \partial_i + m_p^2 c^2 \right) \right] \Phi, \tag{56}$$

where α_{GUP} is the GUP parameter, and m_p is the mass of the scalar particle. After using WKB ansatz, we find the following equation in the leading order of \hbar as follows:

$$\frac{1}{F} (\partial_t S)^2 = G (\partial_r S)^2 + m_p^2 c^2 \left(1 - 2\alpha_{GUP} G (\partial_r S)^2 - 2\alpha_{GUP} m_p^2 c^2 \right). \tag{57}$$

Then, we impose the Hamilton–Jacobi ansatz:

$$S(t, r) = -E t + W(r), \tag{58}$$

and thus Eq. (57) reduces to

$$\frac{1}{F} E^2 = G (W')^2 + m_p^2 c^2 \left(1 - 2\alpha_{GUP} G (W')^2 - 2\alpha_{GUP} m_p^2 c^2 \right). \tag{59}$$

For the above equation, we obtain the outgoing and ingoing (\pm) radial solutions

$$W(r)_\pm = \pm \int \frac{1}{\sqrt{FG}} \frac{\sqrt{E^2 - F \left(m_p^2 c^2 \right) \left(1 - 2m_p^2 c^2 \alpha_{GUP} \right)}}{\sqrt{1 - 2m_p^2 c^2 \alpha_{GUP}}} dr. \tag{60}$$

As the integral becomes zero at the horizon, we use the complex-integral path method to find the solution near the horizon. Namely, we get

$$T_{GUP} = \frac{\kappa}{2\pi} \sqrt{1 - 2m_p^2 c^2 \alpha_{GUP}} = v_0 \frac{f'(r_h)}{2\pi} \sqrt{1 - 2m_p^2 c^2 \alpha_{GUP}}. \tag{61}$$

If we ignore the GUP effect, i.e., $\alpha_{GUP} = 0$, the original Hawking temperature is recovered. Let us note that a similar analysis can be applied to the GUP effects of vector particles, yielding a similar conclusion. Moreover, in Ref. [144], it was shown that the GUP Hawking temperature related to the vector particles is also affected by the nature of particles (i.e. their mass and spin), such that the GUP Hawking temperature would be slightly different from that related to scalar particles. The latter remark might be important for Planck-scale physics. In our case, however, we work in the 1 + 1 EAWD metric; hence, the result will be quite similar. Therefore, in order to see the difference, one needs to consider the problem of quantum tunneling from the 3 + 1 EAWD spacetime.

5 QNMs of particular EAWD spacetime

In this section, we explore the analytical forms of the QNMs of the (1+1)-dimensional EAWD spacetime given as Eq. (38):

$$ds^2 = -N dt^2 + \frac{c^2}{N} dr^2, \tag{62}$$

where $N = c^2 - \bar{v}^2(r)$. Without loss of generality, throughout this section, we use geometric unit system ($c = \hbar = 1$) and focus on a particular position-dependent velocity: $\bar{v}(r) \equiv \bar{f}(r) = e^{-\frac{r}{r_0}}$ [a suitable smooth function satisfying the conditions $\bar{f}(0) = 1$ and $\bar{f}(r \rightarrow \infty) \rightarrow 0$], which represents a velocity that exponentially decreases with r . We want to stress that all results in this section depend on this

particular choice of $\bar{v}(r)$. Because, according to our observations, this is the probably only the case for which one can perform a complete analytical computation of QNMs in the EAWD spacetime. For a discussion of the velocity function of EAWD spacetime, the reader is referred to [4] and references therein.

We first consider the massive KGE:

$$\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\nu} \partial_\nu \varphi) - \mu^2 \varphi = 0, \tag{63}$$

where μ is the mass of the scalar field φ . Choosing the following ansatz for the scalar field

$$\varphi = e^{-i\omega t} \mathcal{R}(r), \tag{64}$$

where ω is the frequency or energy of the flux of scalar particles at spatial infinity, Eq. (63) is shown to be separable in the EAWD background (38). Then, one can reduce the massive KGE to the following radial equation

$$N \partial_r^2 \mathcal{R}(r) + e^{-r} \partial_r \mathcal{R}(r) + \left(\frac{\omega^2 - \mu^2 N}{N} \right) \mathcal{R}(r) = 0. \tag{65}$$

After setting a new variable $z = N = 1 - e^{-r}$, Eq. (65) becomes

$$z(1-z) \partial_z^2 \mathcal{R}(z) + (1-2z) \partial_z \mathcal{R}(z) + \frac{\omega^2 - \mu^2 z}{z(1-z)} \mathcal{R}(z) = 0. \tag{66}$$

In sequel, if one uses the following s-homotopic transformation [79,87,89]

$$\mathcal{R}(z) = z^\alpha (1-z)^\beta \mathcal{F}(z), \tag{67}$$

where

$$\alpha = -i\omega, \tag{68}$$

$$\beta = i\sqrt{\omega^2 - \mu^2}, \tag{69}$$

the radial Equation (66) turns out to be a hypergeometric differential equation [90]:

$$z(1-z) \partial_z^2 \mathcal{F}(z) + [\hat{c} - (\hat{a} + \hat{b} + 1)z] \partial_z \mathcal{F}(z) - \hat{a}\hat{b}\mathcal{F}(z) = 0. \tag{70}$$

The coefficients \hat{a} , \hat{b} , and \hat{c} are given by

$$\hat{a} = \alpha + \beta = i\omega \left(\sqrt{1 - \frac{\mu^2}{\omega^2}} - 1 \right), \tag{71}$$

$$\hat{b} = \alpha + \beta + 1 = i\omega \left(\sqrt{1 - \frac{\mu^2}{\omega^2}} - 1 \right) + 1, \tag{72}$$

$$\hat{c} = 1 + 2\alpha = 1 - 2i\omega. \tag{73}$$

The three regular singular points of Eq. (70) are located at $z = 0$, $z = 1$, and $z = \infty$. There are two independent solutions of Eq. (70) [90]:

$$\mathcal{F}(z) = A_1 F(\widehat{a}, \widehat{b}; \widehat{c}; z) + A_2 z^{1-\widehat{c}} F(1 + \widehat{a} - \widehat{c} + 1, 1 + \widehat{b} - c; 2 - \widehat{c}; z), \tag{74}$$

where A_1, A_2 are constants and $F(\widehat{a}, \widehat{b}; \widehat{c}; z)$ stands for the Gaussian hypergeometric function [90]. Thus, the analytical solution of Eq. (66) is given by

$$\begin{aligned} \mathcal{R}(z) = & A_1 z^{-i\omega} (1 - z)^{i\sqrt{\omega^2 - \mu^2}} F(\widehat{a}, \widehat{b}; \widehat{c}; z) \\ & + A_2 z^{i\omega} (1 - z)^{i\sqrt{\omega^2 - \mu^2}} F(1 + \widehat{a} - \widehat{c} + 1, 1 + \widehat{b} - c; 2 - \widehat{c}; z). \end{aligned}$$

Meanwhile, it is worth noting that setting $\omega = 0$, which corresponds to $\widehat{c} = 1$, two solutions of Eq. (74) become linearly dependent. In this case, the general solution represents a bound state Ref. [91].

In the vicinity of the event horizon ($z \rightarrow 0$), the radial function $\mathcal{R}(z)$ behaves as

$$\mathcal{R}(z) \sim A_1 e^{-i\omega \ln z} + A_2 e^{i\omega \ln z}. \tag{75}$$

Namely, the near horizon form of the scalar field φ reads

$$\varphi \sim A_1 e^{-i\omega(t + \ln z)} + A_2 e^{-i\omega(t - \ln z)}. \tag{76}$$

It is clear from Eq. (76) that the first term represents the ingoing wave while the second term stands for the outgoing wave. For obtaining the QNMs, one should impose the requirement that there exist only ingoing waves at the event horizon. Therefore, to satisfy this condition, we set $A_2 = 0$. Thus, the physically acceptable radial solution for QNMs is given by

$$\mathcal{R}(z) = A_1 z^{-i\omega} (1 - z)^{i\omega\sqrt{1 - \frac{\mu^2}{\omega^2}}} F(\widehat{a}, \widehat{b}; \widehat{c}; z). \tag{77}$$

For matching the near horizon and asymptotic regions, we are interested in the large r behavior ($z \rightarrow 1$) of the solution (77). To this end, we use the linear transformation law $z \rightarrow 1 - z$ for the hypergeometric functions [90]:

$$\begin{aligned} \mathcal{R}(z) = & A_1 z^{-i\omega} (1 - z)^{i\omega\sqrt{1 - \frac{\mu^2}{\omega^2}}} \frac{\Gamma(\widehat{c})\Gamma(\widehat{c} - \widehat{a} - \widehat{b})}{\Gamma(\widehat{c} - \widehat{a})\Gamma(\widehat{c} - \widehat{b})} F(\widehat{a}, \widehat{b}; \widehat{a} + \widehat{b} - \widehat{c} + 1; 1 - z) + \\ & + A_1 z^{-i\omega} (1 - z)^{-i\omega\sqrt{1 - \frac{\mu^2}{\omega^2}}} \frac{\Gamma(\widehat{c})\Gamma(\widehat{a} + \widehat{b} - \widehat{c})}{\Gamma(\widehat{a})\Gamma(\widehat{b})} \\ & F(c - \widehat{a}, \widehat{c} - \widehat{b}; \widehat{c} - \widehat{a} - \widehat{b} + 1; 1 - z). \end{aligned} \tag{78}$$

Therefore, the asymptotic form (around $z = 1$) of the radial solution (77) is given by

$$\begin{aligned} \mathcal{R}(z) \sim & A_1(1-z)^{i\omega\sqrt{1-\frac{\mu^2}{\omega^2}}} \frac{\Gamma(\widehat{c})\Gamma(\widehat{c}-\widehat{a}-\widehat{b})}{\Gamma(\widehat{c}-\widehat{a})\Gamma(\widehat{c}-\widehat{b})} \\ & + A_1(1-z)^{-i\omega\sqrt{1-\frac{\mu^2}{\omega^2}}} \frac{\Gamma(\widehat{c})\Gamma(\widehat{a}+\widehat{b}-\widehat{c})}{\Gamma(\widehat{a})\Gamma(\widehat{b})}. \end{aligned} \quad (79)$$

Correspondingly, the near spatial infinity form of the scalar field becomes

$$\begin{aligned} \varphi \sim & A_1 e^{-i\omega\left[t-\sqrt{1-\frac{\mu^2}{\omega^2}}\ln(1-z)\right]} \frac{\Gamma(\widehat{c})\Gamma(\widehat{c}-\widehat{a}-\widehat{b})}{\Gamma(\widehat{c}-\widehat{a})\Gamma(\widehat{c}-\widehat{b})} \\ & + A_1 e^{-i\omega\left[t+\sqrt{1-\frac{\mu^2}{\omega^2}}\ln(1-z)\right]} \frac{\Gamma(\widehat{c})\Gamma(\widehat{a}+\widehat{b}-\widehat{c})}{\Gamma(\widehat{a})\Gamma(\widehat{b})}. \end{aligned} \quad (80)$$

QNMs impose another requirement that the ingoing waves spontaneously must die out at spatial infinity, which means that only outgoing waves are allowed to survive at the infinity. To distinguish the outgoing and ingoing waves in Eq. (80), we impose the following condition

$$\sqrt{1-\frac{\mu^2}{\omega^2}} \in \mathbb{R} > 0. \quad (81)$$

Thus, A_1 and A_2 become the amplitudes of ingoing and outgoing waves, respectively. However, the boundary conditions of QNM impose that the first term of Eq. (80) must be terminated. This can be performed by the poles of the gamma functions $[\Gamma(\widehat{c}-\widehat{a})$ or $\Gamma(\widehat{c}-\widehat{b})]$ seen in the denominator of the first term of Eq. (80). As is well-known, the gamma functions $\Gamma(y)$ have the poles at $y = -n$ for $n = 0, 1, 2, \dots$. So, the massive scalar waves of the EAWD spacetime's QNMs impose the following restrictions:

$$\widehat{c}-\widehat{a} = -n, \quad \text{or} \quad \widehat{c}-\widehat{b} = -n, \quad (n = 0, 1, 2, \dots) \quad (82)$$

From above, we get two sets of QNMs:

$$\omega_{\text{set1}} = -\frac{i}{2} \left(n - \frac{\mu^2}{n} \right), \quad (83)$$

and

$$\omega_{\text{set2}} = -\frac{i}{2} \left(1 + n - \frac{\mu^2}{1+n} \right). \quad (84)$$

It is worth noting that for the set of Eq. 83, in order to avoid the divergence of the frequency, one should exclude $n = 0$ case and consider $n = 1, 2, 3, \dots$. Similar results were obtained in the QNMs of spin- $\frac{1}{2}$ waves propagating in the geometry of Witten black hole [143]. Having stable QNMs, one must have $Im\omega < 0$. Therefore, when we analyze the obtained QNMs, it can be seen that the first set (83) admits the stable modes when

$$\mu < n \quad (n \geq 1), \quad (85)$$

and the second set (84) is stable if

$$\mu < n + 1 \quad (n \geq 0). \tag{86}$$

On the other hand, one may question the existence of the unstable modes depending on the value of μ . However, we want to draw the reader’s attention to Eq. (81), which allows us to decide the type of wave: ingoing or outgoing. When the obtained sets (83) and (84) [with stability requirements given by: Eqs. (85) and (86)] are used in Eq. (81), we thus have

$$\sqrt{1 - \frac{\mu^2}{\omega^2}} \Big|_{\omega = \omega_{\text{set1}}} = \frac{n^2 + \mu^2}{n^2 - \mu^2} > 0, \quad (\text{for } \mu < n \text{ and } n \geq 1), \tag{87}$$

and

$$\sqrt{1 - \frac{\mu^2}{\omega^2}} \Big|_{\omega = \omega_{\text{set2}}} = \frac{(n + 1)^2 + \mu^2}{(n + 1)^2 - \mu^2} > 0, \quad (\text{for } \mu < n + 1 \text{ and } n \geq 0). \tag{88}$$

One can easily deduce from Eqs. (87) and (88) that only stable QNMs can fulfill the wave-identifier condition (81). Namely, QNMs that do not satisfy conditions Eqs. (85) and (86) for being stable will also not satisfy Eq. (81), which allows us to recognize ingoing and outgoing waves at spatial infinity. In short, the analytical method that we followed encompasses only the stable QNMs. On the other hand, our findings must not be interpreted to mean that there are no unstable waves in the EAWD spacetime; instead they must be studied (including the numerical methods) in more detail. As a matter of fact, it is known that in some geometries there are unstable QNMs (see for instance [92,95]). Moreover, in the limit of highly damped modes (i.e., $n \rightarrow \infty$) the QNMs of the EAWD spacetime become mass independent. The latter remark is also in good agreement with previous studies [85,93–95].

6 Conclusions

In this paper, we have studied the Hawking radiation of the EAWD spacetime in terms of the quantum tunneling method. Our results obtained from the quantum tunneling of massive vector and scalar particles are in accordance with the statistical Hawking temperature, which is simply equal to $\frac{\kappa}{2\pi}$. We have also shown that the Hawking temperature can be decreased when the GUP effects are taken into account. QNM analysis of the particular EAWD spacetime (62) has shown that when this spacetime is perturbed by the massive scalar particles, QNMs based on the wave-identifier condition (81) satisfy the stability conditions (83) and (84). Namely, for the exact solution (75) of the KGE we obtained, the unstable modes do not fulfill Eq. (81): therefore only stable QNMs are taken into account. A similar result was obtained for the QNM frequencies of the Dirac field propagating in the uncharged Witten black hole [143].

In future work, we will extend our analysis to the problem of the greybody factor in the EAWD spacetime. In this way, we plan to find analytical expressions for the absorption cross-section, as well as for the decay-rate for the scalar field in the EAWD spacetime.

Acknowledgements We wish to thank the Editor and anonymous Reviewers for their valuable comments and suggestions. Special thanks to Professor A. López-Ortega for helpful discussions and correspondence. This work was supported by the Chilean FONDECYT Grant No. 3170035 (AÖ). AÖ is grateful to the CERN theory (CERN-TH) division for their hospitality where part of this work was done.

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