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Quantum tunneling of massive spin-1 particles from non-stationary metrics

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Abstract We focus on the HR of massive vector (spin-1) particles tunneling from Schwarzschild BH expressed in the Kruskal–Szekeres and dynamic Lemaitre coordinates. Using the Proca equation together with the Hamilton–Jacobi and the WKB methods, we show that the tunneling rate, and its consequence Hawking temperature are well recovered by the quantum tunneling of the massive vector particles.

Keywords Hawking radiation · Regular coordinates · Proca equation · Massive vector particles · Quantum tunneling

1 Introduction

One prediction of the theory of general relativity (GR) devised by Einstein involves BHs. In principle, a BH is classically defined by an area of space called the "event horizon", where everything is swallowed. Beyond the event horizon, matter and light flow freely, but as soon as the horizon's intangible boundary is crossed, matter and light become trapped. So, a BH is an invisible object (i.e., black), at least classically.

Stephen Hawking's prediction that a BH might not be completely black is unarguably one of the important consequences of the quantum mechanics, when integrated with GR [1–3]. In particular, Hawking proved that a semiclassical BH possesses a characteristic temperature of a thermally distributed radiation spectrum, which is the

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so-called Hawking radiation (HR) [1]. Today, in the literature there exists several derivations of the HR, which are proposed to strengthen this staggering theory (see, for example, [4–14]). Among those methods, the quantum tunneling method (QTM) of Angheben [15] and Padmanabhan [7,16] (with their collaborators) has garnered much attention (see [17] and references therein). QTM employs the complex path integral analysis of Kerner and Mann [18,19] in the HJ formalism, which takes account of the WKB approximation [20]. According to the QTM, a wave propagator that is proportional to $\exp\left(\frac{i}{\hbar}S_0 + S_1 + O(\hbar)\right)$ is applied to the wave equation of the tunneling particle under question. Here, each S denotes the classical action of the trajectory of the particles coming out/in from the horizon.

In particle physics, a vector boson is a boson with the spin-1. In particular, the massive vector bosons [21] i.e., $W^\pm$ and Z particles (force carriers of the weak interaction) play a prominent role in the confirmed Higgs Boson [22]. Nowadays, the detection of a massive photon, which is the so-called *Darklight* [23,24] has become very popular in the experimental physics since it is envisaged to explain the dark matter [25]. Furthermore, in theoretical physics, HR of the massive vector particles in stationary BHs have also attracted much attention (see, for example, [26–38]). However, the number of studies regarding the HR of the spin-1 particles from the non-stationary *regular* metrics is very limited [39], and hence those regular spacetimes deserve more research. Such an extension is one of the goals of the present paper. For this purpose, we consider the PE [26,40,41] in the KS [42,43] and DL [44] coordinates. Next we apply the QTM to the PEs, and obtain a set of differential equations for each coordinate system. Those equations enable us to get a coefficient matrix. After setting the determinant of the coefficient matrix to zero, we get the action S_0, which is the leading order in $\hbar$. Then, we show how one can compute the tunneling rate of the vector particles in the non-stationary metrics, and recover the standard Hawking temperature of the Schwarzschild BH.

The paper is organized as follows: In Sect. 2, we first give a brief introduction about the Schwarzschild spacetime in KS coordinates. Then, a detailed calculation of quantum tunneling of spin-1 particles near the KS horizon is provided. Section 3 is devoted to the computation of the HR of the Schwarzschild BH from the tunneling of the massive vector particles in the DL coordinates. The PE with minimum length effect and conclusions are presented in Sect. 4.

2 Quantum tunneling of massive vector particles from Schwarzschild BH in KS coordinates

The well-known Schwarzschild solution is in general described by the coordinates t and r as follows

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2, \tag{1}$$

where $g_{tt} = (1 - 2M/r)$, $g_{rr} = -(1 - 2M/r)^{-1}$, $g_{\theta\theta} = -r^2$, and $g_{\varphi\varphi} = -r^2\sin^2\theta$. Herein, there is an event horizon at $r = 2M$, so that g_{rr} blows up. On the other side when $r < 2M$, the g_{tt} and g_{rr} exchange their signatures, however the signatures of

$g_{\theta\theta}$ and $g_{\varphi\varphi}$ are not affected. Hence, r becomes "timelike" and t becomes "spacelike" inside the event horizon. One could clear up this "coordinate singularity" problem by introducing the KS coordinates [42,43]:

$$ds^2 = A(-d\tau^2 + dR^2) + r^2(d\theta^2 + B^2 d\varphi^2), \tag{2}$$

where $B = sin(\theta)$ and the metric function A is given by

$$A = \frac{32M^3}{r} e^{-\frac{r}{2M}}. \tag{3}$$

Metric (2) covers the entire spacetime manifold of the maximally extended Schwarzschild solution, and it is well-behaved everywhere outside the physical singularity ($r = 0$). The event horizon in the KS coordinates corresponds to $\tau = \pm R$, and the curvature singularity is located at $\tau^2 - R^2 = 1$. Furthermore, in this coordinate system the Killing vector becomes

$$\xi^\mu = \left[\frac{R}{4M}, \frac{\tau}{4M}, 0, 0\right]. \tag{4}$$

The particle energy of a test particle is given by (in terms of the action S) [45,46]

$$E = -\xi^\mu \partial_\mu S = -\left(\frac{R}{4M}\partial_\tau + \frac{\tau}{4M}\partial_R\right) S. \tag{5}$$

On the other hand, for a curved spacetime, the PE is governed by [26]

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Psi^{\nu\mu}) + \frac{m^2}{\hbar^2}\Psi^\nu = 0, \tag{6}$$

where $\Psi_\nu = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ and m represent the spinor fields [26,27,39] and mass of the spin-1 particle, respectively, and

$$\Psi_{\nu\mu} = \partial_\nu\Psi_\mu - \partial_\mu\Psi_\nu. \tag{7}$$

Using metric (1) in Eq. (6), we obtain the following set of differential equations:

$$-(\hbar ArB)^{-2}[m^2 r^2 AB^2\Psi_0 + \hbar^2 r^2 B^2(-\partial_{RR}\Psi_0 + \partial_{\tau R}\Psi_1)$$
$$-\hbar^2 B^2 A(\partial_{\theta\theta}\Psi_0 - \partial_{\theta\tau}\Psi_2) + \hbar^2 BA(\partial_\theta B)(\partial_\tau\Psi_2 - \partial_\theta\Psi_0)$$
$$-\hbar^2 A(\partial_{\varphi\varphi}\Psi_0 - \partial_{\varphi\tau}\Psi_3)] = 0, \tag{8}$$

$$(\hbar ArB)^{-2}[m^2 Ar^2 B^2\Psi_1 - \hbar^2 r^2 B^2(\partial_{\tau R}\Psi_0 - \partial_{\tau\tau}\Psi_1) - \hbar^2 B^2 A(\partial_{\theta\theta}\Psi_1 - \partial_{\theta R}\Psi_2)$$
$$+\hbar^2 BA(\partial_\theta B)(\partial_R\Psi_2 - \partial_\theta\Psi_1) - \hbar^2 A(\partial_{\varphi\varphi}\Psi_1 - \partial_{\varphi R}\Psi_3)] = 0, \tag{9}$$

$$\left(\hbar\sqrt{A}r^2 B\right)^{-2}[m^2 Ar^2 B^2\Psi_2 - \hbar^2 r^2 B^2(\partial_{\theta\tau}\Psi_0 - \partial_{\tau\tau}\Psi_2)$$
$$-\hbar^2 r^2 B^2(\partial_{\theta R}\Psi_1 - \partial_{RR}\Psi_2) - \hbar^2 A(\partial_{\varphi\varphi}\Psi_2 - \partial_{\theta\varphi}\Psi_3)] = 0, \tag{10}$$

$$\left(\hbar\sqrt{A}r^2B^{\frac{3}{2}}\right)^{-2}[m^2Ar^2B\Psi_3 - \hbar^2r^2B(\partial_{\varphi\tau}\Psi_0 - \partial_{\tau\tau}\Psi_3)$$
$$+ \hbar^2r^2B(\partial_{\varphi R}\Psi_1 - \partial_{RR}\Psi_3) + \hbar^2AB(\partial_{\theta\varphi}\Psi_2 - \partial_{\theta\theta}\Psi_3)$$
$$+ \hbar^2A\,(\partial_\theta B)\,(\partial_\theta\Psi_3 - \partial_\varphi\Psi_2)] = 0. \tag{11}$$

Applying the WKB approximation [39]:

$$\Psi_\nu = c_\nu \exp\left[\frac{i}{\hbar}S_0(\tau, R, \theta, \varphi) + S_1(\tau, R, \theta, \varphi) + O(\hbar)\right], \tag{12}$$

and taking the lowest order of $\hbar$, Eqs. (8–11) become

$$\left(-r^2B^2\,(\partial_R S_0)^2 - A\,(\partial_\varphi S_0)^2 - m^2Ar^2B^2 - AB^2\,(\partial_\theta S_0)^2\right)c_0$$
$$+ c_3A\,(\partial_{\tau\varphi}S_0) + c_1r^2B^2\,(\partial_{R\tau}S_0) + c_2AB^2\,(\partial_{\tau\theta}S_0) = 0, \tag{13}$$
$$r^2B^2\,(\partial_{R\tau}S_0)\,c_0 + c_1\left(-r^2B^2\,(\partial_\tau S_0)^2 + A\,(\partial_\varphi S_0)^2 + m^2Ar^2B^2 + AB^2\,(\partial_\theta S_0)^2\right)$$
$$- c_3A\,(\partial_{R\varphi}S_0) - c_2AB^2\,(\partial_{R\theta}S_0) = 0, \tag{14}$$
$$r^2B^4\,[(\partial_{\tau\theta}S_0)\,c_0 - c_1\,(\partial_{R\theta}S_0)] + c_2\left\{AB^2\,(\partial_\varphi S_0)^2\right.$$
$$\left.- B^4r^2\left[(\partial_\tau S_0)^2 - (\partial_R S_0)^2 - m^2A\right]\right\} - AB^2c_3\,(\partial_{\varphi\theta}S_0) = 0, \tag{15}$$
$$r^2B\left[(\partial_{\tau\varphi}S_0)\,c_0 - c_1\,(\partial_{R\varphi}S_0)\right] - c_2AB\,(\partial_{\varphi\theta}S_0)$$
$$+ Br^2\left[m^2A - (\partial_\tau S_0)^2 + (\partial_R S_0)^2 - \frac{A}{r^2}\,(\partial_\theta S_0)^2\right]c_3 = 0. \tag{16}$$

Now, one can obtain a matrix equation $\mathbb{Z}\,(c_0, c_1, c_2, c_3)^T = 0$ [26,27] (the superscript T means the transition to the transposed vector, and $\mathbb{Z}$ represents a 4×4 matrix) with the following non-zero elements:

$$\mathbb{Z}_{11} = \left[-r^2B^2\,(\partial_R S_0)^2 - A\,(\partial_\varphi S_0)^2 - m^2Ar^2B^2 - AB^2\,(\partial_\theta S_0)^2\right],$$

$$\mathbb{Z}_{12} = \mathbb{Z}_{21} = r^2B^2\,(\partial_{R\tau}S_0),$$

$$\mathbb{Z}_{13} = AB^2\,(\partial_{\tau\theta}S_0), \quad \mathbb{Z}_{31} = r^2B^4\,(\partial_{\tau\theta}S_0),$$

$$\mathbb{Z}_{14} = A\,(\partial_{\tau\varphi}S_0), \quad \mathbb{Z}_{41} = r^2B\,(\partial_{\tau\varphi}S_0),$$

$$\mathbb{Z}_{22} = \left[-r^2B^2\,(\partial_\tau S_0)^2 + A\,(\partial_\varphi S_0)^2 + m^2Ar^2B^2 + AB^2\,(\partial_\theta S_0)^2\right],$$

$$\mathbb{Z}_{23} = -AB^2\,(\partial_{R\theta}S_0), \quad \mathbb{Z}_{32} = -r^2B^4\,(\partial_{R\theta}S_0),$$

$$\mathbb{Z}_{24} = -A\,(\partial_{R\varphi}S_0), \quad \mathbb{Z}_{42} = -r^2B\,(\partial_{R\varphi}S_0),$$

$$\mathbb{Z}_{33} = AB^2\,(\partial_\varphi S_0)^2 - B^4r^2\left[(\partial_\tau S_0)^2 - (\partial_R S_0)^2 - m^2A\right],$$

$$\mathbb{Z}_{34} = -AB^2 \left(\partial_{\varphi\theta} S_0 \right), \quad \mathbb{Z}_{43} = -AB \left(\partial_{\varphi\theta} S_0 \right),$$

$$\mathbb{Z}_{44} = Br^2 \left[m^2 A - (\partial_\tau S_0)^2 + (\partial_R S_0)^2 - \frac{A}{r^2} (\partial_\theta S_0)^2 \right]. \tag{17}$$

Let us consider the following HJ solution for the action:

$$S_0 = Q(\tau, R) + k(\theta) + j\varphi, \tag{18}$$

where j denotes the angular momentum of the massive vector particle. Thus, the determinant of $\mathbb{Z}$-matrix yields

$$\det \mathbb{Z} = -m^2 r^2 A B^3 \left\{ A \left[B^2 (\partial_\theta k)^2 + j^2 \right] + r^2 B^2 \left[m^2 A - (\partial_\tau Q)^2 + (\partial_R Q)^2 \right] \right\}^3. \tag{19}$$

The nontrivial solution for $\partial_R Q$ is obtained by the condition of "$\det \mathbb{Z} = 0$" [26]. Hence, after substituting $\partial_\tau Q = -\frac{4ME}{R} - \frac{\tau}{R} \partial_R Q$ [recall Eq. (5)] into Eq. (19), we obtain

$$\partial_R Q_\pm = \frac{4EM\tau r B \pm R \sqrt{16E^2 M^2 r^2 B^2 - A(R^2 - \tau^2) \left\{ \left[(\partial_\theta k)^2 + m^2 r^2 \right] B^2 + j^2 \right\}}}{(R^2 - \tau^2) Br}. \tag{20}$$

where $+(-)$ corresponds to the outgoing (incoming) massive vector particles. The definite integration of Q is given by

$$Q = \int (\partial_R Q) \, dR + (\partial_\tau Q) \, d\tau. \tag{21}$$

Using the identity $\partial_\tau Q = -\frac{4ME}{R} - \frac{\tau}{R} \partial_R Q$, once again, Eq. (21) can be rewritten as

$$Q = \frac{1}{2} \int \frac{\partial_R Q}{R} d(R^2 - \tau^2) - \frac{4ME}{R} \int d\tau. \tag{22}$$

It is obvious that the second term is real in Eq. (22). However, after inserting Eq. (20) into Eq. (22), we see that the imaginary contribution to the action comes only from the first term since it has pole at the horizon. Thus, the complex path integration method [15,16] for the pole located at the horizon ($R = \tau$) yields

$$Im Q_-|_{horizon} \quad = 0, \tag{23}$$

$$Im Q_+|_{horizon} = 4\pi ME. \tag{24}$$

Therefore, the probabilities of the ingoing/outgoing massive vector particles become

$$\Gamma_{absorption} = e^{-\frac{2}{\hbar} Im Q_{-|horizon}} = 1, \tag{25}$$

$$\Gamma_{emission} = -e^{-\frac{2}{\hbar} Im Q_{+|horizon}} = e^{-8\pi ME}. \tag{26}$$

It is worth noting that the above results are in full agreement with the semiclassical QTM [17], which expects a 100 % chance for the ingoing particles to enter the BH, i.e., $\Gamma_{absorption} = 1$, and thereupon computes the probability of the outgoing (tunneling) particles, $\Gamma_{emission}$.

The tunneling rate is then computed by

$$\Gamma = \frac{\Gamma_{emission}}{\Gamma_{absorption}} = e^{-8\pi ME}. \tag{27}$$

Now, recalling the Boltzmann factor (see for example [17]), $\Gamma = e^{-\beta E} = e^{-8\pi ME}$, where β is the inverse temperature we can recover the original Hawking temperature of Schwarzchild BH:

$$T \equiv T_H = \frac{f'(r_h)}{4\pi} = \frac{1}{8\pi M}. \tag{28}$$

3 Quantum tunneling of massive vector particles from Schwarzchild BH in DL coordinates

In 1933, Georges Lemaitre [44] found a coordinate system $\left(\tilde{\tau}, \tilde{R}, \theta, \varphi\right)$ that removes the coordinate singularity at the Schwarzschild BH is given by

$$ds^2 = -d\tilde{\tau}^2 + \frac{d\tilde{R}^2}{F} + 4M^2 F^2 (d\theta^2 + B^2 d\varphi^2), \tag{29}$$

where

$$F = \left[\frac{3}{4M}(\tilde{R} - \tilde{\tau})\right]^{\frac{2}{3}}. \tag{30}$$

The event horizon in the DL coordinates corresponds to $F = 1$ or $\tilde{R} = \frac{4M}{3} + \tilde{\tau}$. Furthermore, the Killing vector reads

$$\xi^\mu = [1, 1, 0, 0], \tag{31}$$

and it leads to the following particle energy [46]:

$$E = -\xi^\mu \partial_\mu S = -\left(\partial_{\tilde{\tau}} + \partial_{\tilde{R}}\right) S. \tag{32}$$

In this coordinate system, PEs (6) with the ansätz (12) are given by

$$c_0 \left(-\left(\partial_\varphi S_0\right)^2 - 4F^3 M^2 B^2 \left(\partial_{\tilde{R}} S_0\right)^2 - 4m^2 F^2 M^2 B^2 - B^2 \left(\partial_\theta S_0\right)^2\right)$$
$$+ 4c_1 F^3 M^2 B^2 \left(\partial_{\tilde{R}\tilde{\tau}} S_0\right) + c_2 B^2 \left(\partial_{\tilde{\tau}\theta} S_0\right) + c_3 \left(\partial_{\tilde{\tau}\varphi} S_0\right) = 0, \tag{33}$$

$$4F^2 M^2 B^2 c_0 \left(\partial_{\tilde{R}\tilde{\tau}} S_0\right) + c_1 \left(4m^2 F^2 M^2 B^2 - 4F^2 M^2 B^2 \left(\partial_{\tilde{\tau}} S_0\right)^2\right.$$
$$\left. + B^2 \left(\partial_\theta S_0\right)^2 + \left(\partial_\varphi S_0\right)^2\right) - c_2 B^2 \left(\partial_{\tilde{R}\theta} S_0\right) - c_3 \left(\partial_{\tilde{R}\varphi} S_0\right) = 0, \tag{34}$$

$$4F^2 M^2 B^2 c_0 \left(\partial_{\tilde{\tau}\theta} S_0\right) - 4c_1 F^3 M^2 B^2 c_1 \left(\partial_{\tilde{R}\theta} S_0\right) + c_2 \left[4m^2 F^2 M^2 B^2 + \left(\partial_\varphi S_0\right)^2\right.$$
$$\left. - 4F^2 M^2 B^2 \left(\partial_{\tilde{\tau}} S_0\right)^2 + 4F^3 M^2 B^2 \left(\partial_R S_0\right)^2\right] - c_3 \left(\partial_{\varphi\theta} S_0\right) = 0, \tag{35}$$

$$4F^2 M^2 B c_0 \left(\partial_{\tilde{\tau}\varphi} S_0\right) - 4F^3 M^2 B c_1 \left(\partial_{\tilde{R}\varphi} S_0\right) - c_2 B \left(\partial_{\varphi\theta} S_0\right)$$
$$+ \left[4m^2 F^2 M^2 B + B \left(\partial_\theta S_0\right)^2 - 4F^2 M^2 B \left(\partial_{\tilde{\tau}} S_0\right)^2 + 4F^3 M^2 B \left(\partial_{\tilde{R}} S_0\right)^2\right] c_3 = 0. \tag{36}$$

Now, one can read the non-zero elements of the coefficient matrix $\aleph \, (c_0, c_1, c_2, c_3)^T = 0$ ($\aleph$ is another 4×4 matrix) as follows

$$\aleph_{11} = \left[-\left(\partial_\varphi S_0\right)^2 - 4F^3 M^2 B^2 \left(\partial_{\tilde{R}} S_0\right)^2 - 4m^2 F^2 M^2 B^2 - B^2 \left(\partial_\theta S_0\right)^2\right],$$

$$\aleph_{12} = 4F^3 M^2 B^2 \left(\partial_{\tilde{R}\tilde{\tau}} S_0\right), \quad \aleph_{21} = 4F^2 M^2 B^2 \left(\partial_{\tilde{R}\tilde{\tau}} S_0\right),$$

$$\aleph_{13} = B^2 \left(\partial_{\tilde{\tau}\theta} S_0\right), \quad \aleph_{31} = 4F^2 M^2 B^2 \left(\partial_{\tilde{\tau}\theta} S_0\right),$$

$$\aleph_{14} = \left(\partial_{\tilde{\tau}\varphi} S_0\right), \quad \aleph_{41} = 4F^2 M^2 B \left(\partial_{\tilde{\tau}\varphi} S_0\right),$$

$$\aleph_{22} = \left[4m^2 F^2 M^2 B^2 - 4F^2 M^2 B^2 \left(\partial_{\tilde{\tau}} S_0\right)^2 + B^2 \left(\partial_\theta S_0\right)^2 + \left(\partial_\varphi S_0\right)^2\right],$$

$$\aleph_{23} = -B^2 \left(\partial_{\tilde{R}\theta} S_0\right), \quad \aleph_{32} = -4F^3 M^2 B^2 \left(\partial_{\tilde{R}\theta} S_0\right),$$

$$\aleph_{24} = -\left(\partial_{\tilde{R}\varphi} S_0\right), \quad \aleph_{42} = -4F^3 M^2 B \left(\partial_{\tilde{R}\varphi} S_0\right),$$

$$\aleph_{33} = [4m^2 F^2 M^2 B^2 + \left(\partial_\varphi S_0\right)^2 - 4F^2 M^2 B^2 \left(\partial_{\tilde{\tau}} S_0\right)^2 + 4F^3 M^2 B^2 \left(\partial_R S_0\right)^2],$$

$$\aleph_{34} = -\left(\partial_{\varphi\theta} S_0\right), \quad \aleph_{43} = -B \left(\partial_{\varphi\theta} S_0\right),$$

$$\aleph_{44} = \left[4m^2 F^2 M^2 B + B \left(\partial_\theta S_0\right)^2 - 4F^2 M^2 B \left(\partial_{\tilde{\tau}} S_0\right)^2 + 4F^3 M^2 B \left(\partial_{\tilde{R}} S_0\right)^2\right]. \tag{37}$$

Inserting the following ansätz for S_0

$$S_0 = \widetilde{Q}(\tilde{\tau}, \tilde{R}) + k(\theta) + j\varphi, \tag{38}$$

into Eq. (37), and subsequently using the energy condition (32), namely:

$$\partial_{\tilde{\tau}} \widetilde{Q} = -(E + \partial_{\tilde{R}} \widetilde{Q}), \tag{39}$$

we get solutions for $\partial_{\tilde{R}}\widetilde{Q}$ from $det\aleph = 0$:

$$det\aleph = -\frac{M^2 F^2 B m^2}{1024} \left\{ B^2 (\partial_\theta k)^2 + j^2 + 4M^2 F^2 B^2 \left[m^2 + F \left(\partial_{\tilde{R}}\widetilde{Q} \right)^2 \right. \right.$$
$$\left. \left. - \left(E + \partial_{\tilde{R}}\widetilde{Q} \right)^2 \right] \right\}^3 = 0, \tag{40}$$

as follows

$$\partial_{\tilde{R}}\widetilde{Q}_\pm = \frac{EMBF \pm \sqrt{E^2 M^2 F^3 B^2 - (F-1)\left[m^2 F^2 M^2 B^2 + \frac{1}{4} B^2 (\partial_\theta k)^2 + \frac{j^2}{4} \right]}}{(F-1) FBM}. \tag{41}$$

Using the energy expression (39) in the definite integration of $\widetilde{Q}$:

$$\widetilde{Q} = \int \partial_{\tilde{R}}\widetilde{Q} d\tilde{R} + \partial_{\tilde{\tau}}\widetilde{Q} d\tilde{\tau}, \tag{42}$$

we obtain

$$\widetilde{Q} = \int \partial_{\tilde{R}}\widetilde{Q} d\left(\tilde{R} - \tilde{\tau} \right) - E \int d\tilde{\tau},$$
$$= 2M \int \frac{\partial_{\tilde{R}}\widetilde{Q}}{\sqrt{F}} dF - E \int d\tilde{\tau}, \tag{43}$$

where $dF = \frac{1}{2M}\sqrt{F}d(\tilde{R} - \tilde{\tau})$ [see Eq. (30)]. It is clear that the second integral of (43) results in real values, which means that it does not give any contribution to the imaginary part of the action. However, after substituting Eq. (41) into Eq. (43), one can see that the first integral of Eq. (43) has a pole at the horizon ($F = 1$), and evaluating it around the pole yields

$$Im\, Q_+|_{horizon} = 4\pi ME, \tag{44}$$

and trivially

$$Im\, Q_-|_{horizon} = 0. \tag{45}$$

The above results are fully consistent with Eqs. (23) and (24). Consequently, they admit the same tunneling rate given in Eq. (27). In short, using the quantum tunneling of the massive vector particles in the DL coordinate system, we have managed to rederive the Hawking temperature ($T_H = \frac{1}{8\pi M}$) of the Schwarzchild BH.

4 Conclusion

In this paper, we have used the PE (6) in order to compute the HR of the massive vector particles tunneling from the Schwarzchild BH given in two different (KS and DL) regular dynamic coordinate systems. In addition to the HJ and the WKB approximation methods, particle energy definitions played crucial role in our computations. The

original Hawking temperature of the Schwarzschild BH is impeccably obtained in the both coordinate systems. Thus, we have shown that HR is independent of the selected coordinate system. The latter remark was also highlighted in [47].

Finally, we plan to extend our present study to the HR of the massive spin-1 particles that experience the minimum length effect [48–51], which is governed by the GUP (generalized uncertainty principle) [52–58]. Such a study requires the following modification in the PE [59]:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[\left(1 - \frac{l_p^2}{3} \partial_\mu^2 \right) \sqrt{-g} \Phi^{\nu\mu} \right] + \frac{m^2}{\hbar^2} \Phi^\nu = 0, \tag{46}$$

where

$$\Phi_{\nu\mu} = \partial_\nu \Phi_\mu - \partial_\mu \Phi_\nu - \partial_\nu \frac{l_p^2}{3} \partial_\mu^2 \Phi_\mu + \partial_\mu \frac{l_p^2}{3} \partial_\nu^2 \Phi_\nu, \tag{47}$$

in which l_p denotes the Planck length. This problem may reveal more information compared to the present case. This is going to be our next study in the near future.

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