



Evolving topologically deformed wormholes supported in the dark matter halo

Ali Övgün^a

Physics Department, Eastern Mediterranean University, via Mersin 10, 99628 Famagusta, North Cyprus, Turkey

Received: 27 May 2021 / Accepted: 16 September 2021

© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract In this paper, we construct an evolving wormhole in the dark matter halo. This work is relevant since matter has two components: (i) cosmological part (only time dependent) and (ii) wormhole part (only space dependent). In order to implement this, we use the Chaplygin gas as an equation of state for the cosmic part and Navarro–Frenk–White dark matter density profile as well as Thomas–Fermi profile in order to form a dark wormhole. The flare-out condition of wormhole is also satisfied by violating the null energy condition for some specific values of quantities. Furthermore, we reveal more interesting results regarding how a topological deformation parameter α affects the evolving wormhole sourced with some dark matter models based on the physically motivated shape function.

1 Introduction

A wormhole is a theoretical passage through space-time, which connects different points and creates a shortcut for traveling in a space-time. Einstein and Rosen initially elaborated on wormholes in 1935 using the theory of general relativity. Subsequently, the main contributions on traversable wormholes were done by Morris, Thorne, and Yurtsever in the 1980s [1, 2]. However, since then, no one has discovered a wormhole until now, it is completely based on theoretical research. Understanding the mysteriousness of the nature of the universe is a puzzle for humanity.

The existence of wormhole geometries is required to utilize the exotic matter, which is one of the most debatable issues. In order to minimize the presence of exotic matter, one is required to use modified theories or extra sources. An alternative approach can be the utilization of thin-shell formalism, which was initially proposed by Visser. Moreover, another important problem is to find a stable wormhole against perturbations. A singularity-free system identifies a stable state and it prevents the wormhole from collapsing. Recently, the studies of wormhole solutions have gained great interest [3–70].

On the other hand, theoretical and observational cosmology struggles to resolve the source of inflation in the primitive universe and also the present cosmic accelerated expansion. Recent experiments show that some enigmatic force may cause the accelerated expansion, which

^a e-mail: ali.ovgun@emu.edu.tr (corresponding author)
URL: <http://www.aovgun.com>

is known as the dark energy. Besides, it also highlights the ambiguous power of cosmology [71, 72].

Naturally, we assume that a wormhole can lead to an apparent failure of locality on the background space-time in the primitive universe. If there was a wormhole in the early universe, it would be inflated into at least the size of a human to allow time travel, with reference to this, the early Roman studied the enlargement of the wormhole by inflation [4]. It was proposed that amidst the inflation era, the wormhole had inflated. Furthermore, Kim extended Roman’s idea and studied the cosmological properties in the Friedmann–Robertson–Walker cosmologies with a traversable wormhole [3]. He has divided the content of matter into two parts: (i) the cosmological part (only time dependent) and (ii) the wormhole part (only space dependent). Subsequently, he studied the behavior of the scale factor and the wormhole shape function in this context. After these seminal papers, the study of exact solutions under assorted scenarios was extensively studied in order to understand the sophisticated picture of cosmic evolution and wormhole. After the creation of this predicament, Cataldo et al., in a subsequent paper [29–32], studied various models of an FRW-like cosmologies using the wormholes in different dimensions. He argued that it is possible to find normal matter wormhole in the universe, and the evolving wormhole metric can cause the acceleration of the universe.

In this paper, our main aim is to study evolving topologically deformed wormhole supported in dark matter halo, especially the Navarro–Frenk–White (NFW) profile [74], which is a spatial mass distribution of dark matter as well as Thomas–Fermi (TF) profile, on the other hand, the cosmic part sourced with the Chaplygin gas (CG) [6]. Different from the previous studies, we investigate the effect of the topological deformation parameter α on the evolving wormhole.

The remainder of this paper is organized as follows: In Sect. 2, we study the evolution of the universe using the CG gas in the topologically deformed FRW space-time. Then, in Sect. 3, we construct a dynamic traversable wormhole by solving Einstein field equations in the background of the topologically deformed FRW space-time. In Sect. 4, we will discuss the results.

2 Topologically deformed Wormhole embedded in FRW cosmology

In this section, we first consider the space-time metric representing a dynamic traversable wormhole in a FRW universe as follows:

$$ds^2 = -e^{2\Phi(r)}dt^2 + R(t)^2 \left(\frac{dr^2}{1-kr^2-\frac{b(r)}{r}} + r^2\alpha^2d\Omega^2 \right), \tag{2.1}$$

where b is function of r and known as a shape function, α is the solid angle deficit ($0 < \alpha < 1$) as well as $\phi(r)$ is the lapse function. $R(t)$ is the scale factor of the universe. Moreover, k is the curvature of space-time with values $+1, 0, -1$. Note that $R(t) \rightarrow$ constant the static Morris–Thorne wormhole is recovered.

To avoid the form of event horizon, these conditions should be satisfied: $(1 - kr^2 - \frac{b(r)}{r}) > 0$, and $b(r_0) = r_0$ at the throat. Solving the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ for the above metric and $\phi(r) = 0$, we obtain the nonzero components of the Einstein tensor with energy density ρ , radial pressure P^r and lateral pressure P^l reduce to

$$\rho = \frac{1}{8\pi} \left[\frac{3(\dot{R}^2 + k)}{R^2} + \frac{1}{R^2} \frac{b'}{r^2} + \frac{-\alpha^2 + 1}{r^2 \alpha^2 R^2} \right], \tag{2.2}$$

$$P^r = \frac{1}{8\pi} \left[-2 \frac{\ddot{R}}{R} - \frac{(\dot{R}^2 + k)}{R^2} - \frac{1}{R^2} \frac{b}{r^3} - \frac{(\alpha - 1)(\alpha + 1)(-kr^3 + (R^2 + 1)r - b)}{r^2 R^2 \alpha^2 (kr^3 + b - r)} \right], \tag{2.3}$$

$$P^t = \frac{1}{8\pi} \left[-2 \frac{\ddot{R}}{R} - \frac{(\dot{R}^2 + k)}{R^2} + \frac{1}{2R^2} \left(\frac{b}{r^3} - \frac{b'}{r^2} \right) \right]. \tag{2.4}$$

Note that radial tension $\tau = -P^r$ is equal to the negative radial pressure and $H = \dot{R}/R$ is the cosmological Hubble parameter. Moreover an overdot stands for differentiation with respect to t . Then, we use the following ansatz for matter parts to separate field equations in two parts [3]

$$R^2(t)\rho(r, t) = R^2(t)\rho_c(t) + \rho_w(r), \tag{2.5}$$

$$R^2(t)P^r(r, t) = R^2(t)P_c(t) + P_w^r(r), \tag{2.6}$$

$$R^2(t)P^t(r, t) = R^2(t)P_c(t) + P_w^t(r). \tag{2.7}$$

Note that above equations depend on R^2 , which shows that the wormhole affects the curvature. Furthermore, we use the subscripts c and w to refer the cosmic and wormhole parts. With the help of ansatz equations (2.5)–(2.7), we rewrite the Einstein equations in two parts as follows

$$R^2 \left[8\pi\rho_c - 3 \left(\frac{\dot{R}}{R} \right)^2 - \frac{3k}{R^2} \right] = \frac{b'}{r^2} - 8\pi\rho_w + \frac{-\alpha^2 + 1}{r^2 \alpha^2} = l, \tag{2.8}$$

$$R^2 \left[8\pi P_c + 2 \frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} \right] = -\frac{b}{r^3} - 8\pi P_w^r - \frac{(\alpha - 1)(\alpha + 1)(-kr^3 + rR^2 - b + r)}{r^2 \alpha^2 (kr^3 + b - r)} = m, \tag{2.9}$$

$$R^2 \left[8\pi P_c + 2 \frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} \right] = \frac{b - rb'}{2r^3} - 8\pi P_w^t = m. \tag{2.10}$$

However, a new term arises corresponding to a linear potential. Note that l and m are constants. The separation constants l and m are also determined by using the wormhole matter distribution. On the other hand, for the cosmological part, the conservation laws $T_{\nu;\mu}^\mu = 0$ give us the following equations:

$$\dot{\rho}_c + 3H(\rho_c + P_c) = \frac{q}{k} \dot{R}R^{-3}, \tag{2.11}$$

where $q = l + 3m$. To investigate the cosmological part, we use the equation of state (EOS) of CG as follows [6]:

$$P_c = \frac{-A}{\rho_c}, \tag{2.12}$$

where A is constant. The solution of Eq. (2.11) for $q = 0$ by using Eq. (2.12) gives us the cosmic density for Chaplygin gas ρ_c as follows

$$\rho_c = \sqrt{B + \frac{A}{R^6}}, \tag{2.13}$$

where B is integration constant. For small R ($\rho_c \sim R^6 \ll A/B$), this solution reduces to $\rho_c \sim \frac{\sqrt{A}}{R^3}$. On the other hand, for large value of R , $\rho_c \sim \sqrt{B}$, $\tau_c \sim -\sqrt{B}$. Using Eqs. (2.8)–(2.10) with $l = -m = 0$, we find the following FRW equations:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi\rho_c - \frac{k}{R^2}, \tag{2.14}$$

$$-2\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 = 8\pi P_c + \frac{k}{R^2}. \tag{2.15}$$

Then, using the cosmological matter distribution $\rho_c \sim \frac{\sqrt{A}}{R^3}$, and $k = 0$, we find the scale factor $R(t)$ for small value of R

$$R(t) = \frac{(3c + 2\sqrt{6\pi}\sqrt[4]{At})^{2/3}}{2^{2/3}}, \tag{2.16}$$

which is for a universe dominated by dust-like matter. We also calculate the scale factor $R(t)$ for the large value of R , using the $\rho_c \sim \sqrt{B}$:

$$R(t) = e^{2\sqrt{\frac{2\pi}{3}}\sqrt[4]{B}t}, \tag{2.17}$$

which is for an empty universe with a cosmological constant.

In the next section, we will check the possibility of evolving topologically deformed wormhole supported various dark matter halos.

3 Construction of evolving topologically deformed wormhole with Dark Matter Halo

Now, we construct the evolving topologically deformed wormholes supported in the dark matter halo. For this purpose, we use the wormhole part of Eqs. (2.8)–(2.10) by choosing $l = -3m$

$$\frac{b'}{r^2} - 8\pi\rho_w + \frac{-\alpha^2 + 1}{r^2\alpha^2} = -3m, \tag{3.1}$$

$$-\frac{b}{r^3} - 8\pi P_w^r - \frac{(\alpha - 1)(\alpha + 1)(-kr^3 + rR^2 - b + r)}{r^2\alpha^2(kr^3 + b - r)} = m, \tag{3.2}$$

$$\frac{b - rb'}{2r^3} - 8\pi P_w^t = m. \tag{3.3}$$

From Eqs. (3.1)–(3.3), we obtain that $\rho_w + P_w^r + 2P_w^t = 0$. We choose the one of the pressure in the form of $P_w^r = \omega_r\rho_w$, then the lateral pressure becomes barotropic $P_w^t = -(1 + \omega_r)\rho_w/2$. To check the maintenance of the wormhole with time, we study the embedding space in a flat three dimensional Euclidean space using the $t = \text{const}$ with equatorial plane $\theta = \pi/2$:

$$ds^2 = d\bar{z}^2 + d\bar{r}^2 + \bar{r}^2\alpha^2d\varphi^2. \tag{3.4}$$

Taking the relations: $\bar{r} = R(t)r$, $d\bar{r}^2 = R^2(t)dr^2$, and constant time slice [4], we get

$$ds^2 = \frac{R^2(t)dr^2}{1 - kr^2 - \frac{b(r)}{r}} + R^2(t)r^2\alpha^2 d\varphi^2. \tag{3.5}$$

Then, it is obtained that [1]:

$$\frac{d\bar{z}}{d\bar{r}} = \pm \left(\frac{k\bar{r}^3 + \bar{b}}{\bar{r} - k\bar{r}^3 - \bar{b}(\bar{r})} \right)^{1/2} = \frac{dz}{dr}. \tag{3.6}$$

Equation (3.6) reduces to

$$\begin{aligned} \bar{z}(\bar{r}) &= \pm \int d\bar{r} \left(\frac{k\bar{r}^3 + \bar{b}}{\bar{r} - k\bar{r}^3 - \bar{b}(\bar{r})} \right)^{1/2} = \pm R(t) \int \left(\frac{kr^3 + b}{r - kr^3 - b(r)} \right)^{1/2} dr \\ &= \pm R(t)z(r). \end{aligned} \tag{3.7}$$

The flare-out condition for the evolving wormhole become: $d^2\bar{r}(\bar{z})/d\bar{z}^2 > 0$ near the throat:

$$\frac{d^2\bar{r}(\bar{z})}{d\bar{z}^2} = \frac{1}{R(t)} (\bar{b} - \bar{b}'\bar{r} - 2k\bar{r}^3) = \frac{1}{R(t)} \frac{d^2r(z)}{dz^2} > 0. \tag{3.8}$$

where $\bar{b}'(\bar{r}) = d\bar{b}/d\bar{r} = b'(r) = db/dr$. For the constant $R(t) = 1$, the flare-out condition reduces to the static wormhole at throat:

$$\frac{d^2\bar{r}(\bar{z})}{d\bar{z}^2} = (\bar{b} - \bar{b}'\bar{r} - 2k\bar{r}^3) > 0. \tag{3.9}$$

3.1 NFW dark matter profile

NFW density profile is used to study dark matter halo for galaxies and clusters. Here, we find a wormhole solution in the NFW dark matter density profile, by using ρ_w : [74]

$$\rho_w = \frac{\rho_s}{\frac{r}{R_s} \left(1 + \frac{r}{R_s} \right)^2}, \tag{3.10}$$

where R_s is known as radius of characteristic scale and ρ_s is the dark matter density when the dark matter halo collapses. Using Eq. (3.10) within Eq. (3.1), and using the boundary condition for the wormhole $b(r_0) = r_0$, the shape function $b(r)$ is calculated as follows:

$$\begin{aligned} b(r) &= \frac{8\pi R_s^4 \rho_s (r_0 - r)}{(R_s + r_0)(R_s + r)} - 8\pi R_s^3 \rho_s \ln(R_s + r_0) + 8\pi R_s^3 \rho_s \ln(R_s + r) \\ &\quad + \frac{\alpha^2 m (r_0^3 - r^3) + r_0 - r}{\alpha^2} + r. \end{aligned} \tag{3.11}$$

For the condition of $m = 0$, shape function $b(r)$ reduces to

$$\begin{aligned} b(r) &= \frac{8\pi R_s^4 \rho_s (r_0 - r)}{(R_s + r_0)(R_s + r)} \\ &\quad - 8\pi R_s^3 \rho_s \ln(R_s + r_0) + 8\pi R_s^3 \rho_s \ln(R_s + r) + \frac{r_0 - r}{\alpha^2} + r. \end{aligned} \tag{3.12}$$

For constant $R = 1$, we check the maintenance of the shape of the traversable wormhole, using the following flare-out equation which must be positive and derived by violating the

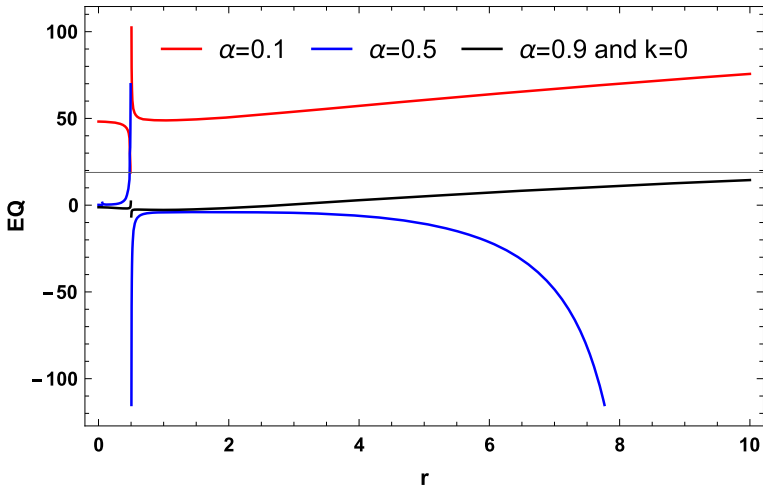


Fig. 1 Plot of the flare-out equation EQ versus r which must be positive

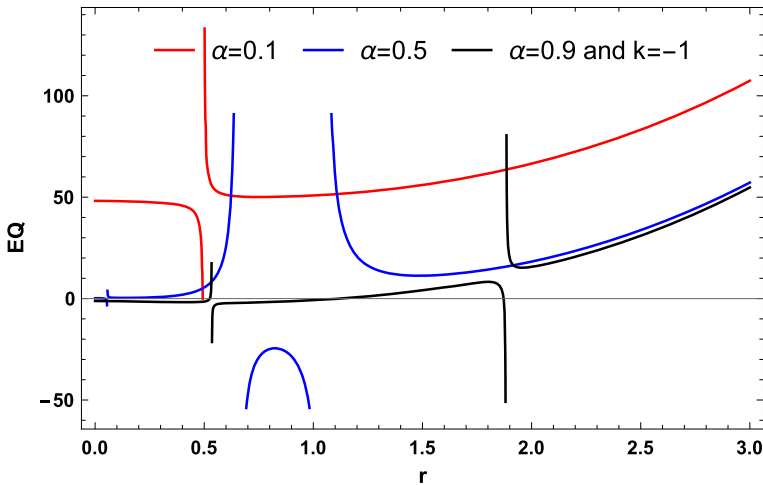


Fig. 2 Plot of the flare-out equation EQ versus r which must be positive

null energy condition (NEC) ($\rho + P^r \geq 0$) via Eqs. (2.2)–(2.3)

$$EQ = -\frac{r^2}{\alpha^2 (b(r) + kr^3 - r)} + \frac{r^2}{b(r) + kr^3 - r} + b(r) - rb'(r) - 2kr^3 > 0, \quad (3.13)$$

which is plotted in Figs. 1 and 2, showing that Eq. (3.13) is not satisfied on the whole domain and bounded. Moreover, the evolving topologically deformed wormhole may be created with normal matter, however, Eq. (3.13) will become negative and its lifetime can be said that limited. It is clearly shown in figures that the topological deformation parameter α change the stable region for the wormhole.

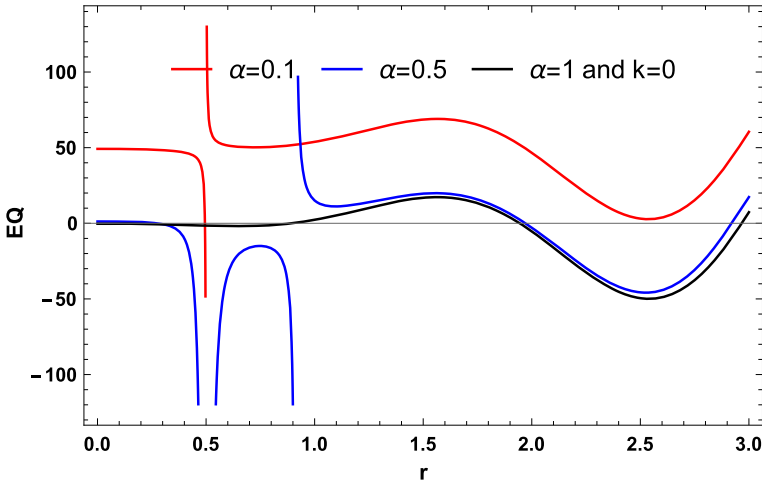


Fig. 3 Plot of the flare-out equation EQ versus r which must be positive

3.2 Thomas–Fermi (TF) profile

Here, we construct a wormhole solution in the model of Bose–Einstein condensation (BEC) dark matter (DM) which has more advantages on the small scales of galaxies. For this model, density profile is [73]

$$\rho_{TF} = \rho_s \frac{\sin(kr)}{kr} \tag{3.14}$$

where ρ_s is the center density of Bose–Einstein condensation dark matter halo, and $k = \pi/R$ is the radius where the dark matter pressure and density vanish. Note that The BEC-DM model supports much less dark matter density in the center regions of galaxies than the NFW profile.

$$b(r) = \frac{8\rho_s R^2 \left(R \left(\sin\left(\frac{\pi r}{R}\right) - \sin\left(\frac{\pi r_0}{R}\right) \right) + \pi r_0 \cos\left(\frac{\pi r_0}{R}\right) - \pi r \cos\left(\frac{\pi r}{R}\right) \right)}{\pi^2} + r_0^3 m + \frac{r_0 - r}{\alpha^2} - mr^3 + r. \tag{3.15}$$

For constant R , we check the maintenance of the shape of the traversable wormhole, using the following flare-out equation which must be positive

$$EQ = -\frac{r^2}{\alpha^2 (b(r) + kr^3 - r)} + \frac{r^2}{b(r) + kr^3 - r} + b(r) - rb'(r) - 2kr^3 > 0. \tag{3.16}$$

Note that it is plotted in Figs. 3 and 4 showing that Eq. (3.16) is not satisfied on the whole domain and bounded. Moreover, the evolving topologically deformed wormhole may be created with normal matter, however, Eq. (3.16) will become negative and its lifetime can be said that limited.

3.3 Wormhole supported Chaplygin gas and NFW dark matter profile

Here, we combine the Chaplygin gas for the large value of R with the NFW dark matter profile $\rho_w = \rho_c + \rho_{NFW}$ to obtain the shape function of the evolving topologically deformed

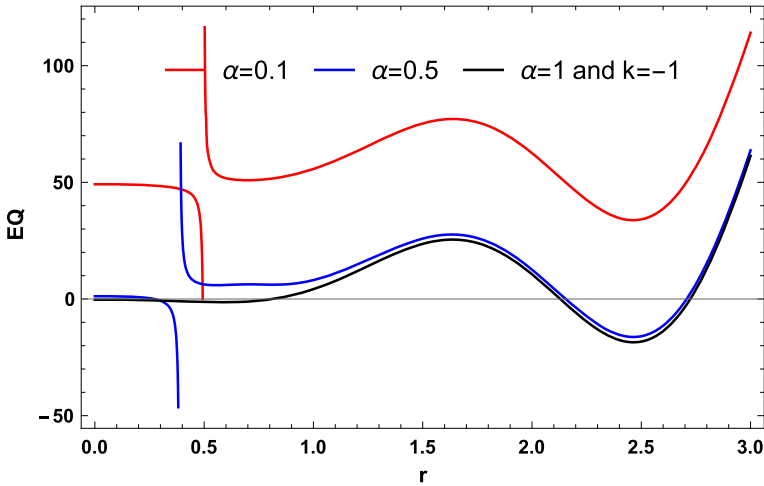


Fig. 4 Plot of the flare-out equation EQ versus r which must be positive

wormholes. The new combined energy density becomes:

$$\rho_w = \frac{\rho_s}{\frac{r}{R_s} \left(1 + \frac{r}{R_s}\right)^2} + \sqrt{B}, \tag{3.17}$$

then, the shape function is calculated as follows:

$$\begin{aligned} b(r) = & -\frac{8\pi R_s^4 \rho_s}{R_s + r_0} - \frac{8\pi R_s^4 \rho_s \log(R_s + r_0)}{R_s + r_0} + \frac{8\pi R_s^4 \rho_s}{R_s + r} - \frac{8\pi R_s^3 \rho_s r_0 \log(R_s + r_0)}{R_s + r_0} \\ & + 8\pi R_s^3 \rho_s \log(R_s + r) - \frac{8\pi \sqrt{B} r_0^4}{3(R_s + r_0)} \\ & - \frac{8\pi R_s \sqrt{B} r_0^3}{3(R_s + r_0)} + \frac{r_0^4 m}{R_s + r_0} + \frac{R_s r_0^3 m}{R_s + r_0} + \frac{r_0^2}{\alpha^2 (R_s + r_0)} + \frac{R_s r_0}{\alpha^2 (R_s + r_0)} \\ & + \frac{8}{3} \pi \sqrt{B} r^3 - m r^3 - \frac{r}{\alpha^2} + r. \end{aligned} \tag{3.18}$$

For constant R , we check the maintenance of the shape of the traversable wormhole, using the following flare-out equation which must be positive

$$EQ = -\frac{r^2}{\alpha^2 (b(r) + kr^3 - r)} + \frac{r^2}{b(r) + kr^3 - r} + b(r) - r b'(r) - 2kr^3 > 0, \tag{3.19}$$

which is plotted in Figs. 5 and 6 showing that Eq. (3.19) is not satisfied on the whole domain and bounded. Moreover, the evolving topologically deformed wormhole may be created with normal matter, however, Eq. (3.19) will become negative and its lifetime can be said that limited.

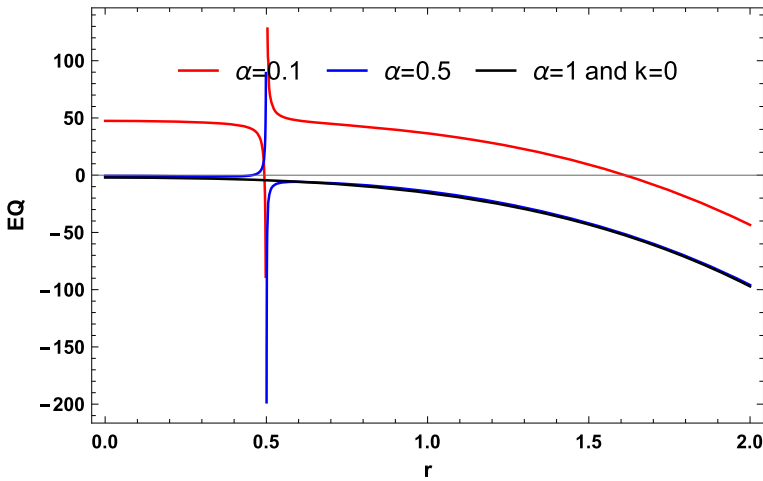


Fig. 5 Plot of the flare-out equation EQ versus r which must be positive

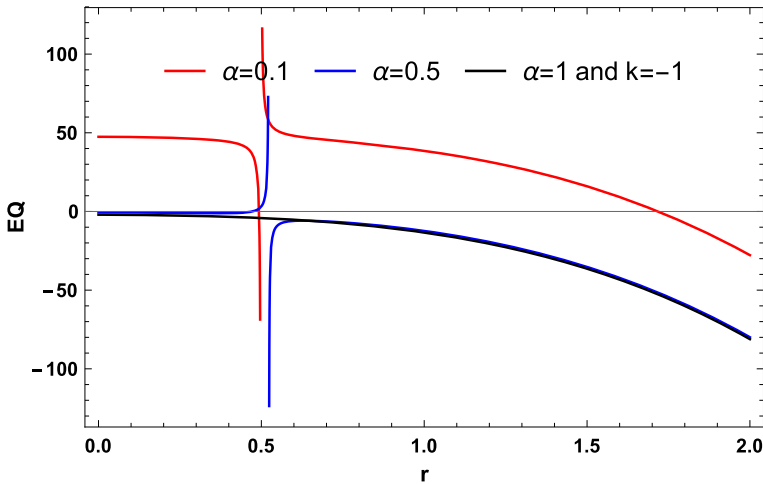


Fig. 6 Plot of the flare-out equation EQ versus r which must be positive

4 Conclusions

In this paper, we have obtained the evolving topologically deformed wormhole supported in the dark matter halo and check its behavior of the wormhole in the inflation era. The solution of the field equations given in Eqs. (2.2)–(2.4) has been divided into two independent systems using the ansatz given in Eqs. (2.5)–(2.7). One of the Einstein field equations is for the static gravitational field on the other hand; second one is for the time-dependent gravitational field. We have used the first part of the field equation to construct evolving topologically deformed wormhole by obtaining the shape function $b(r)$ for different models of dark matter halo, on the other hand, second equation which is the time-dependent part (Friedmann-like with curvature k), have been used to study cosmological models within Chaplygin gas by obtaining the scale factor $R(t)$. To do so, we have used the Chaplygin gas as the equation

of states in the cosmological part and three different dark matter models for the evolving topologically deformed wormhole part such as NFW dark matter profile, Thomas–Fermi profile, and combination of the Chaplygin gas with NFW dark matter profile. Then, the shape functions of the evolving wormhole are derived for these three models separately, and we have plotted their NEC conditions to show their maintenance. The effect of topological deformation parameter α to present in the evolving wormhole metric is also explored. During the inflation era, the amount of the exotic matter needed is decreasing for the evolving wormhole as compared to the static wormhole. In Figs. 1 and 2, we have plotted the EQ versus ‘ r ’ with different values of the topological deformation parameter α to show the effect of the α on the NEC. Second, we have constructed the evolving topologically deformed wormhole using the TF profile for BEC-DM model. Figures 3 and 4 show that NEC is not satisfied on the whole domain and bounded. Moreover, the evolving topologically deformed wormhole may be created with normal matter, however, Eq. (3.19) will become negative and its lifetime can be said that limited. Last, we have combined the Chaplygin gas with NFW dark matter to obtain the shape function of the evolving topologically deformed wormhole. Here, we have shown that NEC is in general satisfied for the range of parameters shown in Figs. 5 and 6. Our results have shown that the topological deformation parameter α affects the stable region for the wormhole.

References

1. M.S. Morris, K.S. Thorne, *Am. J. Phys.* **56**, 395 (1988)
2. M.S. Morris, K.S. Thorne, U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988)
3. S.W. Kim, *Phys. Rev D* **53**, 6889 (1996)
4. T.A. Roman, *Phys. Rev. D* **47**, 1370 (1993)
5. V.P. Frolov, I. Novikov, *Phys. Rev. D* **48**, 1607 (1993)
6. V. Gorini, A. Kamenshchik, U. Moschella and V. Pasquier, [arXiv:gr-qc/0403062](https://arxiv.org/abs/gr-qc/0403062) [gr-qc]
7. M. Visser, D. Hochberg, *Annals Israel Phys. Soc.* **13**, 249 (1997)
8. S.W. Kim, S.P. Kim, *Phys. Rev. D* **58**, 087703 (1998)
9. B. Mirza, M. Eshaghi, S. Dehdashti, *Int. J. Mod. Phys. D* **15**, 1217 (2006)
10. A.V.B. Arellano, F.S.N. Lobo, *Class. Quant. Grav.* **23**, 5811 (2006)
11. F.S.N. Lobo, J. Martínez-Asencio, G.J. Olmo, D. Rubiera-García, *Phys. Rev. D* **90**(2), 024033 (2014)
12. F.S.N. Lobo, G.J. Olmo, D. Rubiera-García, *Eur. Phys. J. C* **74**(6), 2924 (2014)
13. A.V.B. Arellano, N. Breton, R. García-Salcedo, *Gen. Rel. Grav.* **41**, 2561 (2009)
14. P.E. Kashargin, S.V. Sushkov, *Phys. Rev. D* **78**, 064071 (2008)
15. H. Maeda, T. Harada, B.J. Carr, *Phys. Rev. D* **79**, 044034 (2009)
16. V. Dzhunushaliev, V. Folomeev, *Gen. Rel. Grav.* **42**, 1889 (2010)
17. R.A. El-Nabulsi, *Chin. Phys. Lett.* **26**, 090401 (2009)
18. E.I. Guendelman, A. Kaganovich, E. Nissimov, S. Pacheva, *Int. J. Mod. Phys. A* **25**, 1405 (2010)
19. E.I. Guendelman, *Gen. Rel. Grav.* **23**, 1415 (1991)
20. I. Bochićchio, V. Faraoni, *Phys. Rev. D* **82**, 044040 (2010)
21. S.N. Sajadi, N. Riazi, *Prog. Theor. Phys.* **126**, 753 (2011)
22. I. Bochićchio, E. Laserra, *Int. J. Theor. Phys.* **52**, 3721 (2013)
23. M.K. Zangeneh, F.S.N. Lobo, N. Riazi, *Phys. Rev. D* **90**(2), 024072 (2014)
24. Y. Tomikawa, T. Shiromizu, K. Izumi, *Phys. Rev. D* **90**(12), 126001 (2014)
25. S. Najafi, T. Rostami, S. Jalalzadeh, *Annals Phys.* **354**, 288 (2015)
26. Y. Tomikawa, K. Izumi, T. Shiromizu, *Phys. Rev. D* **91**(10), 104008 (2015)
27. S. Bahamonde, M. Jamil, P. Pavlovic, M. Sossich, *Phys. Rev. D* **94**(4), 044041 (2016)
28. S.W. Kim, *Phys. Lett. B* **780**, 174 (2018)
29. M. Cataldo, P. Labrana, S. del Campo, J. Crisostomo, P. Salgado, *Phys. Rev. D* **78**, 104006 (2008)
30. M. Cataldo, S. del Campo, P. Manning, P. Salgado, *Phys. Rev. D* **79**, 024005 (2009)
31. M. Cataldo, F. Arostica, S. Bahamonde, *Phys. Rev. D* **88**(4), 047502 (2013)
32. M. Cataldo, L. Liempi, P. Rodríguez, *Eur. Phys. J. C* **77**(11), 748 (2017)
33. M. Sharif, H.I. Fatima, *Gen. Rel. Grav.* **48**(11), 148 (2016)

34. M. Sharif, I. Nawazish, *Annals Phys.* **389**, 283 (2018)
35. M. Sharif, K. Nazir, *Annals Phys.* **393**, 145 (2018)
36. K.A. Bronnikov, S.W. Kim, *Phys. Rev. D* **67**, 064027 (2003)
37. K.A. Bronnikov, R.A. Konoplya, A. Zhidenko, *Phys. Rev. D* **86**, 024028 (2012)
38. K.A. Bronnikov, J.P.S. Lemos, *Phys. Rev. D* **79**, 104019 (2009)
39. S. Bhattacharya, S. Chakraborty, *Eur. Phys. J. C* **77**(8), 558 (2017)
40. A. Mishra, S. Chakraborty, *Eur. Phys. J. C* **78**(5), 374 (2018)
41. M. Halilsoy, A. Övgün, S.H. Mazharimousavi, *Eur. Phys. J. C* **74**, 2796 (2014)
42. I. Sakalli, A. Övgün, *Eur. Phys. J. Plus* **130**(6), 110 (2015)
43. A. Övgün and I. Sakalli, *Theor. Math. Phys.* **190**, no. 1, 120 (2017) [*Theor. Mat. Fiz.* **190**, no. 1, 138 (2017)]
44. I. Sakalli, A. Övgün, *Astrophys. Space Sci.* **359**(1), 32 (2015)
45. A. Övgün, M. Halilsoy, *Astrophys. Space Sci.* **361**(7), 214 (2016)
46. A. Övgün, *Eur. Phys. J. Plus* **131**(11), 389 (2016)
47. A. Övgün, K. Jusufi, *Adv. High Energy Phys.* **2017**, 1215254 (2017)
48. K. Jusufi, A. Övgün, *Mod. Phys. Lett. A* **32**(07), 1750047 (2017)
49. A. Övgün, I.G. Salako, *Mod. Phys. Lett. A* **32**(23), 1750119 (2017)
50. A. Övgün, K. Jusufi, *Eur. Phys. J. Plus* **132**(12), 543 (2017)
51. K. Jusufi, A. Övgün, A. Banerjee, *Phys. Rev. D* **96**(8), 084036 (2017)
52. K. Jusufi, A. Övgün, *Phys. Rev. D* **97**(2), 024042 (2018)
53. M. G. Richarte, I. G. Salako, J. P. Morais Graca, H. Moradpour and A. Övgün, *Phys. Rev. D* **96**, no. 8, 084022 (2017)
54. K. Jusufi, A. Övgün, A. Banerjee, I. Sakalli, *Eur. Phys. J. Plus* **134**(9), 428 (2019)
55. R. Myrzakulov, L. Sebastiani, S. Vagnozzi, S. Zerbini, *Class. Quant. Grav.* **33**(12), 125005 (2016)
56. P. Bhar, F. Rahaman, *Eur. Phys. J. C* **74**(12), 3213 (2014)
57. F. Rahaman, P. Salucci, P.K.F. Kuhfittig, S. Ray, M. Rahaman, *Annals Phys.* **350**, 561 (2014)
58. A. Övgün, *Turk. J. Phys.* **44**(5), 465–471 (2020)
59. A. Övgün, *Universe* **5**(5), 115 (2019)
60. A. Övgün, G. Gyulchev, K. Jusufi, *Annals Phys.* **406**, 152–172 (2019)
61. A. Övgün, *Phys. Rev. D* **98**(4), 044033 (2018)
62. A. Övgün, K. Jusufi, I. Sakalli, *Phys. Rev. D* **99**(2), 024042 (2019)
63. F. Rahaman, N. Paul, A. Banerjee, S.S. De, S. Ray, A.A. Usmani, *Eur. Phys. J. C* **76**(5), 246 (2016)
64. Z. Xu, M. Tang, G. Cao, S.N. Zhang, *Eur. Phys. J. C* **80**(1), 70 (2020)
65. M. Kord Zangeneh, F.S.N. Lobo, H. Moradpour, *Phys. Dark Univ.* **31**, 100779 (2021)
66. F.S.N. Lobo, A. Simpson, M. Visser, *Phys. Rev. D* **101**(12), 124035 (2020)
67. I. Ayuso, F.S.N. Lobo, J.P. Mimoso, *Phys. Rev. D* **103**(4), 044018 (2021)
68. M. Kord Zangeneh, F.S.N. Lobo, *Eur. Phys. J. C* **81**, no.4, 285 (2021)
69. R. Garattini, F.S.N. Lobo, *Fundam. Theor. Phys.* **189**, 111–135 (2017)
70. P. K. F. Kuhfittig, [[arXiv:2103.02519](https://arxiv.org/abs/2103.02519)] [gr-qc]
71. A. Övgün, G. Leon, J. Magana, K. Jusufi, *Eur. Phys. J. C* **78**(6), 462 (2018)
72. A. Övgün, *Eur. Phys. J. C* **77**(2), 105 (2017)
73. C.G. Boehmer, T. Harko, *JCAP* **06**, 025 (2007)
74. J.F. Navarro, C.S. Frenk, S.D.M. White, *Astrophys. J.* **462**, 563 (1996)