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Weak deflection angle by asymptotically flat black holes in Horndeski theory using Gauss–Bonnet theorem

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The principal objective of this project is to investigate the gravitational lensing by asymptotically flat black holes in the framework of Horndeski theory in weak field limits. To achieve this objective, we utilize the Gauss–Bonnet theorem to the optical geometry of asymptotically flat black holes and apply the Gibbons–Werner technique to achieve the deflection angle of photons in weak field limits. Subsequently, we manifest the influence of plasma medium on deflection of photons by asymptotically flat black holes in the context of Horndeski theory. We also examine the graphical impact of deflection angle on asymptotically flat black holes in the background of Horndeski theory in plasma medium as well as non-plasma medium.

Keywords: Weak gravitational lensing; deflection of light; black hole; deflection angle; Horndeski theory; Gauss–Bonnet theorem.

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1. Introduction

The anecdote of a falling apple fostering Newtonian gravity has been imparted on generations since the year 1666. For centuries since then, physicists have believed

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that the gravitational force is related to the ratio of the product of the interacting masses and the square of their separation through a proportionality constant. This "Gravitational" constant was given an accurate value via the Cavendish experiment. In 1910s, Einstein's general theory of relativity transpired suggesting a finite, spherical universe in which the Gravitational constant, G was discerned to exhibit a dependence on mass distribution and size of the universe, so as to account for the inertial forces [1].

A decade later, Hubble's observations confirmed the Big Bang Theory, essentially establishing that the universe was expanding, and potentially infinite. The extent of the universe that can be observed is limited to a maximum distance determined by the time that light takes to reach Earth from the observable edge. In other words, mass distribution and the size of the universe change with time, and, hence, so does the gravitational constant, G [2, 3]. This discrepancy led to the speculation that the effect of G could rather be a scalar field, than a constant number. According to Einstein's formulation, the so-called metric field contains the influence of gravity, and mathematically known as a tensor. Therefore, the idea of consolidating a scalar field due to mass distribution with the metric is called as the scalar-tensor theory [4–6].

One such generic gravitational scalar-tensor theory is the Horndeski theory [7]. Defined for a four-dimensional spacetime, the scalar field is incorporated as a new degree of freedom to formulate the Lagrangian of the system, begetting second-order field equations of motion. This notion is inspired from the Lovelock theory of gravity: relaxing Lovelock's assumptions not only exercises the scalar–tensor theory, but also extends Einstein's theory of gravity.

The kinetic term of the Lagrangian characterizes the quadratic derivative of the field and depends on the Lovelock tensor, which is proportional to the Einstein tensor, $G_{\mu\nu}$, encompassing non-minimal coupling between the scalar field, φ and curvature [8]. Hence, the action principle for a 4D spacetime ($n = 4$) becomes [9, 10]

$$I[g_{\mu\nu}, \varphi] = \int \sqrt{-g}\, d^4x \left[k(R - 2\Lambda) - \frac{1}{2}(\alpha g_{\mu\nu} - \eta G_{\mu\nu})\nabla^\mu \varphi \nabla^\nu \varphi \right], \qquad (1)$$

where $k \equiv 1/16\pi G$. Here, the first term accounts for the scalar field with non-minimal coupling for matter owing to the Ricci scalar, R, and the second term accounts for the Einstein–Hilbert action for gravity owing to the cosmological constant, Λ. Values of the parameters α and η are governed by the positive energy density of matter field.

The above equation is the reduced form of the action analyzed with standard matter, matter fields and non-standard scalar field [10]: implementing assumptions — the geometry is static, spherically symmetric and homogeneous with the scalar and metric fields obeying this symmetry in an asymptotically flat spacetime — renders this equation as the limiting case of Horndeski theory. Equation (1) is the foundation of the ensuing work.

When the gravitational force of a massive object bends the trajectory of light that originated from a distant object behind it, gravitational lensing occurs. This is a consequence of general relativity used to understand the universe, galaxies, dark energy and dark matter [1]. Various authors studied the gravitational lensing by black holes, wormholes, cosmic strings and other objects since the first gravitational lensing observation by Eddington [11–37].

In 2008, Gibbons and Werner cleverly showed that there is an alternative way to obtain the weak deflection angle for asymptotically flat optical spacetimes using the Gauss–Bonnet theorem [38]. Later, Werner managed to derive the weak deflection angle of stationary spacetimes using the same [39]. Note that both considered the source and the observer to be placed at asymptotic regions. Next, Ishihara *et al.* showed that it is also possible to use this method for finite-distances (large impact parameter cases) [40]. Then, Crisnejo and Gallo showed that the plasma medium deflects photons [41]. For more recent works, one can see [42–82].

Multi-messenger astronomy constrained the scalar-tensor theories substantially through the detection of GW170817. The arrival times of the gravitational waves and their electromagnetic counterparts from the NGC 4993 galaxy were observed to have fluctuated by less than a minute when two neutron stars spiralling each other ultimately merged. The speed of the gravitational wave is seen to be affected when the scalar field is coupled to curvature. To be coherent with these observations, quintic and quartic models are neglected restricting our calculations to linear observables. Note that Horndeski theories have a serious flaw, related to their primordial tensor spectrum, namely, the gravitational wave speed is not equal to unity. Theories of this sort are problematic. For example, the detection of GW170817 eliminates any late-universe application of Horndeski theory [83]. These theories can be amended by using a new framework developed in [84], firstly developed in [85] and improved by [86–88]. The results inferred by these studies are, however, beyond the scope of this work.

In this paper, we intend to study the deflection angle of the black holes governed by Horndeski theory using the Gauss–Bonnet theorem to test the validity of the modified gravity theory. To compare, we consider the idea of the deflection angle of massive particles in a plasma medium from a black hole. Our main aim is to check the effects of Horndeski theory on weak deflection angle.

This paper is organized as follows. Section 2 reviews some basics on asymptotically flat black holes, computes the Gaussian optical curvature and calculates the deflection angle using GBT. In Sec. 3, we calculate the deflection angle in plasma medium, followed by concluding remarks in the last section.

2. Calculation of Photon Lensing for Asymptotically Flat Black Holes

When the action comprises of a cosmological term, a new asymptotically locally flat black hole can be found. Here, the kinetic term (constructed with Einstein tensor)

of the scalar field alone is considered to yield the matter term taking $\alpha = 0$ in the action, which reduces the latter to

$$I[g_{\mu\nu}, \varphi] = \int \sqrt{-g} d^4 x [k(R - 2\Lambda) + \frac{\eta}{2} G_{\mu\nu} \nabla^\mu \varphi \nabla^\nu \varphi]. \tag{2}$$

In [84], the authors have obtained an equation applying the slow-roll conditions to incorporate the consequences due to the experimental findings of GW170717, further simplified by [88]: the action appears to acquire an extra term equal to $\varphi \mathcal{G}$ where $\mathcal{G} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$.

Setting the integration constant that emanated from the first integral of the field equation to zero, the following metric defines a solution of the system for $K \neq 0$ [9]:

$$ds^2 = -H(r)dt^2 + \frac{15(\Lambda r^2 - 2K)^2}{K} \frac{dr^2}{H(r)} + r^2 d\Sigma_{K,2}^2, \tag{3}$$

where $d\Sigma_{K,2}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and

$$H(r) = (60K^2 - 20\Lambda K r^2 + 3\Lambda^2 r^4) - \frac{\mu}{r}. \tag{4}$$

If Λ disappears, the scalar field vanishes reducing the solution to that of the topological Schwarzschild solution in a flat spacetime, representing a black hole only in a spherically symmetrical scenario [9]. By taking $\Lambda = 0$,

$$ds^2 = -H(r)dt^2 + \frac{60K}{H(r)}dr^2 + r^2 d\Sigma_{K,2}^2, \tag{5}$$

and

$$H(r) = 60K^2 - \frac{\mu}{r}. \tag{6}$$

Here, μ is the integration constant and can be explicate as the black hole mass. Now to acquire the null geodesics ($ds^2 = 0$), the black hole optical spacetime in equatorial plane $\theta = \pi/2$ is written as

$$dt^2 = \frac{60K}{H(r)^2}dr^2 + \frac{r^2}{H(r)}d\varphi^2, \tag{7}$$

along with optical metric $\hat{g}_{ab}^{\text{opt}} = \frac{g_{ab}}{(-g_{tt})}$, according to the Fermat principle, the geodesics are spatial rays of light. The Gaussian optical curvature can be determined by allowing use of the description for the two-dimensional (2D) optical metric given in Eq. (7)

$$\mathcal{K} = \frac{R_{\text{icciScalar}}}{2}, \tag{8}$$

in which R for optical metric is the Ricci scalar. Following the computation of non-zero Christoffel symbols, we obtain the following equation particularly for the Gaussian optical curvature of the optical metric

$$\mathcal{K} = -\frac{K\mu}{r^3} + \frac{\mu^2}{80\,Kr^4} + \mathcal{O}(r^{-5}). \tag{9}$$

Let us recall the GBT for a 2D manifold. In this regard, we consider a regular domain D_R aligned by 2D surface S with Riemannian metric $\hat{g}_{ij}$, along with its

piece-wise smooth boundary $\partial D_R = \gamma_g \cup C_R$, then GBT permits a connection among the geometry and topology in terms of the subsequent relation [38]

$$\int\int_{D_R} \mathcal{K} dS + \oint_{\partial D_R} \hat{k} d\sigma + \sum_j \tilde{\theta}_j = 2\pi\mathcal{X}(D_R), \tag{10}$$

where $\mathcal{K}$ is the Gaussian optical curvature, $\tilde{\theta}_j$ is the exterior angle at the jth vertex and σ is the line element along the boundary D_R. Let γ be a smooth curve in the same domain. Thus, $\dot{\gamma}$ comes to be the unit speed vector [38]. It is well known that for regular domain the Euler characteristic $\mathcal{X}_{D_R} = 1$, while $\hat{k}$ is termed as a geodesic curvature and is defined as

$$\hat{k} = g^{\text{opt}}(\nabla_{\dot{\gamma}}\dot{\gamma}, \ddot{\gamma}), \tag{11}$$

having the unit speed condition $g^{\text{opt}}(\dot{\gamma}, \dot{\gamma}) = 1$, where $\ddot{\gamma}$ is the unit acceleration vector perpendicular to $\dot{\gamma}$. In the case of $R \to \infty$, the respective jump angles are taken as $\pi/2$ (in short, the sum of angles corresponding to the observer and the source: $\tilde{\theta}_O + \tilde{\theta}_S \to \pi$). Using the fact that the geodesic curvature offers no contribution i.e. $\hat{k}(\gamma_{\tilde{g}}) = 0$, we shall pursue a contribution by the virtue of the curve C_R computed as

$$\hat{k}(C_R) = |\nabla_{\dot{C}_R}\dot{C}_R|. \tag{12}$$

Let us consider $C_R := r(\varphi) = R = \text{const}$, while R endows the distance from the coordinate origin. The radial component of the geodesic curvature is stated as

$$(\nabla_{\dot{C}_R}\dot{C}_R)^r = \dot{C}_R^\varphi(\partial_\varphi \dot{C}_R^\varphi) + \Gamma_{\varphi\varphi}^r(\dot{C}_R^\varphi)^2. \tag{13}$$

Using the above equation, we note that the first term vanishes, then the second term can be obtained using the unit speed condition. Then, $\hat{k}$ is calculated as $\lim_{R\to\infty} \hat{k}(C_R) = \lim_{R\to\infty} |\nabla_{\dot{C}_R}\dot{C}_R| \to \frac{1}{R}$. We take the large limits of the radial distance, and find: $\lim_{R\to\infty} dt \to R\,d\varphi$. Hence the deflection angle can be calculated in the form [38]

$$\Theta = -\int_0^\pi \int_{b/\sin\varphi}^\infty \mathcal{K} dS, \tag{14}$$

where b is the impact parameter, a dimensionless quantity that endorses the straight line approximation in which the light ray is assumed to be expressed as $r = b/\sin\varphi$ at zeroth order in the weak deflection limits [38]. This equation inscribes the global impact on the lensing of particles on account of the fact that one has to integrate over the optical domain of integration outside the enclosed mass. Now, by using Eq. (9) into Eq. (14), we obtain the weak deflection angle for flat black holes in Horndeski theory to be

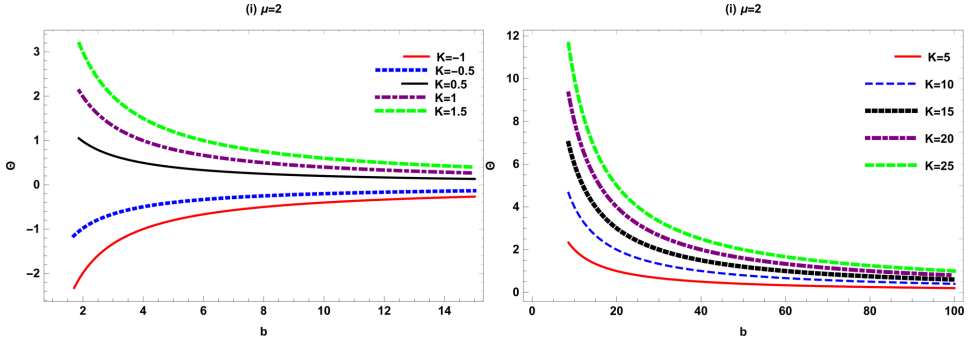$$\Theta = \frac{2K\mu}{b} + \frac{\mu^2\pi}{320Kb^2} + \mathcal{O}(\mu^3). \tag{15}$$

Fig. 1. Θ versus b.

2.1. *Graphical analysis*

This segment is dedicated to review the impact of deflection angle Θ on asymptotically flat black holes graphically and to illustrate the physical eminence of these graphs to examine the influence of curvature constant K and impact parameter b on the deflection angle by analyzing the stable and unstable state of the black hole.

2.1.1. *Deflection angle versus impact parameter*

For $\mu = 2$, the deflection angle is Θ plotted against the impact parameter b for different values of the curvature constant K in Fig. 1.

- Figure 1 demonstrates the influence of Θ with respect to b for different values of K. One can examine that for small b deflection angle increases but as b increases, the deflection angle decreases for fixed μ. So, for stable behavior we choose the domain $b \in [1, 15]$.

  Figure 1(a) illustrates graphically the impact of Θ with respect to b by varying K. For negative K, we obtain locally hyperbolic behavior but for $K = 0$ the behavior is locally flat. If there is small change in the variation of K, deflection angle is exponentially decreasing.
- Figure 1(b) shows that with the increase of K, Θ decreases gradually as it tends to positive infinity. We obtain physical stable behavior just for $K \geq 0.5$.

3. Photon Lensing in a Plasma Medium

In this section, we investigate the effect of plasma on photon lensing by asymptotically flat black hole in Horndeski theory. The refractive index for a flat black hole is stated as follows [41]:

$$n(r) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2} H(r)}, \tag{16}$$

then, the corresponding optical metric illustrated as

$$d\tilde{\sigma}^2 = g_{jk}^{\text{opt}} dx^j dx^k = \frac{n^2(r)}{H(r)} \left[\frac{60K}{H(r)} dr^2 + r^2 d\varphi^2 \right], \tag{17}$$

where, the metric function $H(r)$ in the optical metric is given by

$$H(r) = 60K^2 - \frac{\mu}{r}. \tag{18}$$

The corresponding optical Gaussian curvature is calculated by using Eq. (8) to be

$$\mathcal{K} = -\frac{\mu K}{r^3} + \frac{\mu^2}{80Kr^4} + \frac{90\mu K^3}{r^3} \frac{\omega_e^2}{\omega_\infty^2} - \frac{9\mu^2 K}{4r^4} \frac{\omega_e^2}{\omega_\infty^2} - \frac{3\mu K}{r^4}$$

$$\times (60K^2 r - \mu) \frac{\omega_e^2}{\omega_\infty^2} + \frac{\mu^3}{80Kr^5} \frac{\omega_e^2}{\omega_\infty^2} + \frac{3\mu^2(60K^2 r - \mu)}{80Kr^5} \frac{\omega_e^2}{\omega_\infty^2}. \tag{19}$$

Then the geodesic curvature approaches 1 for R goes to ∞ as

$$\lim_{R \to \infty} \hat{k}_g \frac{d\tilde{\sigma}}{d\varphi} \bigg|_{C_R} = 1. \tag{20}$$

Using the straight line approximation given by $r = b/\sin\varphi$ as $R \to \infty$, GBT can be stated as [41]

$$\lim_{R \to \infty} \int_0^{\pi+\Theta} \left[\hat{k}_g \frac{d\tilde{\sigma}}{d\varphi} \right] \bigg|_{C_R} d\varphi = \pi - \lim_{R \to \infty} \int_0^{\pi} \int_{b/\sin\varphi}^{R} \mathcal{K} dS. \tag{21}$$

After simplification, we obtain

$$\Theta \simeq \frac{\mu^2 \pi}{320 b^2 K} + \frac{\mu^3}{90 b^3 K} \frac{\omega_e^2}{\omega_\infty^2} + \frac{2\mu K}{b} - \frac{3\mu^2 K\pi}{4b^2} \frac{\omega_e^2}{\omega_\infty^2}$$

$$+ \frac{180\mu K^3}{b} \frac{\omega_e^2}{\omega_\infty^2} + \mathcal{O}(\mu^4, K^4). \tag{22}$$

The above results show that the photon rays are moving in a medium of homogeneous plasma.

3.1. *Graphical analysis*

This section is focused to investigate the graphical effect of deflection angle Θ on asymptotically flat black holes in a plasma medium. Further, we exemplify the physical implications of these graphs to analyze the effect of curvature constant K, $\frac{\omega_e}{\omega_\infty}$ and impact parameter b on deflection angle by analyzing the stable and unstable state of black hole.
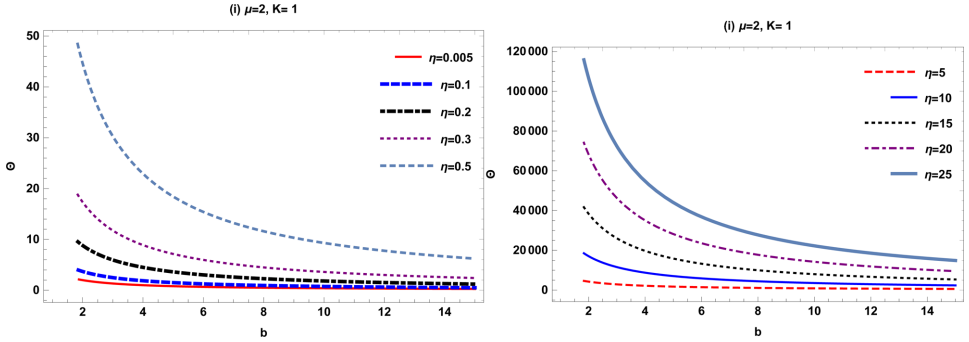
Fig. 2. Relation between Θ and impact parameter b.

3.1.1. *Deflection angle versus impact parameter b*

This section offers the examination of deflection angle Θ with respect to impact parameter b for different ranges of curvature constant K, $\frac{\omega_e}{\omega_\infty}$, for fixed $\mu = 2$. For simplicity, here, we suppose $\frac{\omega_e}{\omega_\infty} = \eta$.

- Figure 2 depicts the influence of Θ with respect to b for varying η and for fixed $\mu = 2$ and $K = 1$.

  (1) Figure 2(a) represents the behavior of Θ with respect to b for small variation of η. For $\eta = 0.005$ the deflection angle decreases, it is observed that the behavior is same for $\eta = 0.005 \rightarrow 0.01$ and $0.01 \rightarrow 0.1$ there is small change in deflection angle but greater than 0.1 the deflection angle actually increases.

  (2) Figure 2(b) shows that the deflection angle gradually increases by increasing η.

4. Conclusion

In this paper, we accomplished an extensive analysis of deflection angle of light by asymptotically flat black hole in the background of Horndeski theory in weak field approximation. In this regard, we employ the optical geometry of asymptotically flat black hole in Horndeski theory. Thenceforth, we have utilized the GBT by using straight line approximation and computed the deflection angle procured by the leading order terms. The obtained deflection angle is evaluated by integrating a domain outside the impact parameter, that depict the globally impact of gravitational lensing. Additionally, we have found the deflection angle of photon lensing for asymptotically flat black hole in a plasma medium. Also, we have analyzed the influence of the impact parameter, the curvature constant and the plasma term on the deflection angle of the photon lensing by asymptotically flat black hole in the context of Horndeski theory graphically. We infer that the proposed deflection angle increases by decreasing the impact parameter, the mass term μ is found to decrease the deflection angle, and increasing the curvature constant is seen to decrease the

deflection angle gradually. Moreover, if we disregard the impact of plasma medium, $(\frac{\omega_e}{\omega_\infty} \to 0)$, in the following equation:

$$\Theta \simeq \frac{\mu^2\pi}{320b^2K} + \frac{\mu^3}{90b^3K}\frac{\omega_e^2}{\omega_\infty^2} + \frac{2\mu K}{b} - \frac{3\mu^2K\pi}{4b^2}\frac{\omega_e^2}{\omega_\infty^2}$$

$$+ \frac{180\mu K^3}{b}\frac{\omega_e^2}{\omega_\infty^2} + \mathcal{O}(\mu^4, K^4), \tag{23}$$

we obtain the weak deflection angle for non-plasma medium case:

$$\Theta = \frac{2K\mu}{b} + \frac{\mu^2\pi}{320Kb^2} + \mathcal{O}(\mu^3). \tag{24}$$

The observations that follow from the Horndeski theory and its mathematical implications include determining observables such as angular positions, separation, magnification and fluxes: a case study of astrophysical applications for Sagittarius A* and M87 can be found in [11]. Additionally, distinct researches include linear Horndeski theories to study dark energy and gravitational waves [89], and Horndeski gravity to study dark matter [90]. Our results can be extrapolated to correct for quantum effects while determining the observables, thus, increasing precision, however, the resulting calculations to attain it are beyond the scope of this work. Furthermore, it will be interesting to study the weak gravitational lensing of black holes in MAXWELL $f(R)$ gravity theories as well as in fourth order gravity theories [91–93] and non-Minimal Horndeski-like theories (the gravitational wave speed can be equal to unity in natural units, when an appropriate Gauss–Bonnet term is added in the action) using the GBT in future [84].

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