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Weak gravitational lensing by Einstein-nonlinear-Maxwell–Yukawa black hole

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In this paper, we analyze the weak gravitational lensing in the context of Einstein-nonlinear-Maxwell–Yukawa black hole. To this desire, we derive the deflection angle of light by Einstein-nonlinear-Maxwell–Yukawa black hole using the Gibbons and Werner method. For this purpose, we obtain the Gaussian curvature and apply the Gauss–Bonnet theorem to find the deflection angle of Einstein-nonlinear-Maxwell–Yukawa black hole in weak field limits. Moreover, we derive the deflection angle of light in the influence of plasma medium. We also analyze the graphical behavior of deflection angle by Einstein-nonlinear-Maxwell–Yukawa black hole in the presence of plasma as well as non-plasma medium.

Keywords: Weak gravitational lensing; Einstein-nonlinear-Maxwell–Yukawa black holes; deflection angle; Gauss–Bonnet theorem.

Mathematics Subject Classification 2020: 83C15, 83C57

1. Introduction

In our universe the black holes (BHs) are essential components and when stars die, they become very dense objects which is the most important discovery of astrophysics. To test the fundamental laws of universe, BHs provide a golden opportunity. For example, the neutron star merges and the gravitational waves from BHs have been discovered recently. Therefore, the gravitational lensing of BHs has attracted incredible attention in the last few decades, essentially because of the solid proof of supermassive BHs at the focal point of galaxies [1, 2]. By utilizing the gravitational lensing, we can simplify the study of BHs, which is a general investigative method

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for acquiring the time delays of the images, magnification and positions. Darwin [3] was the first who analyzed the Schwarzschild geometry. After that in 1985, Herschel [4] published his similar article and after that many authors [5, 6], extended this geometry to general spherically symmetric BHs and Reissner–Nordstrom geometry. Kerr BHs [7–10] were also discussed for acquiring the time delays of the images, magnification and positions by using gravitational lensing. Many modifications have been done through deflection of light and modification in the reference of nonlinear electrodynamics. NLE [11] has been studied through the various alternative gravity theories [12]. Classically, a black hole contains singularity and also horizon because in general theory of relativity, the spacetime singularities create a dozen of problems which are physical as well as mathematical. Therefore, many people use various methods to remove these singularities from the BHs like modified gravity, effect of quantum gravity and NLE. Bozza [13] discussed these topics in his recent paper in detail, which also includes observational prospectus and additional references [14–20].

The main focus of this paper is to calculate the deflection angle of photon by using GBT. The aspect of deflection of light has been widely studied in different astrophysical system in the influence of strong as well as weak gravitational lensing [18–21]. Gibbons and Werner [22] played a very important role in this area and highlighted the significance of topological effect on the deflection of angle by utilizing the GBT and optical geometry. Additionally, they have calculated deflection angle for the Schwarzschild BHs, which is different from the other methods, assuming that the light ray by taking the domain outside, where the mass is closed in the given area on space, is strongly related to the lensing effect. Recently, Gibbons and Werner (GW) [22] computed the deflection angle by applying GBT as follows:

$$\sigma = -\iint_{D\infty} \mathcal{K} dS.$$

Here, $\mathcal{K}$ is Gaussian curvature and dS represents the surface element. Afterwards, Werner [23] extended this method and obtained the deflection of light by Kerr–Newman BHs by applying Nazims's method for Rander–Finsler metric. Recently, deflection angle for a asymptotically flat and spherically symmetric spacetime by using the finite distance from an observer to a light source has been calculated by Ishihara *et al.* [24–26]. Moreover, Asada *et al.* [27] in stationary axisymmetric spacetime calculated the weak gravitational lensing. In all these methods, by using GBT, they have calculated the weak gravitational lensing which shows the global properties. The study of gravitational lensing in the presence of plasma medium has been discussed in a number of cases. Initially, Bisnovatyi–kogan [28, 29] showed that the gravitational deflection in plasma is different from vacuum deflection angle due to precise properties of plasma. Recently, Gallo and Crisnejo [30] discussed the deflection angle of photon in a plasma medium.

Javed *et al.* [31] examined the Einstein–Maxwell theory (EMT) for hairy BH in the context of weak gravitational lensing with a non-minimally coupled dilaton.

After that, there is a lot of literature [32–70] related to the investigation of weak gravitational lensing through the GBT method on various BHs, cosmic strings and wormholes. In this paper, we study the weak gravitational lensing by Einstein-nonlinear-Maxwell–Yukawa BHs.

This paper is composed as follows: In Sec. 2, we concisely review about Einstein-nonlinear-Maxwell–Yukawa black hole. In Sec. 3, by using the Gauss–Bonnet theorem, we compute the deflection angle of Einstein-nonlinear-Maxwell–Yukawa black hole. In Sec. 4, we work to investigate the deflection angle in the influence of plasma medium. We also demonstrate the graphical effect of deflection angle in the context of Einstein-nonlinear-Maxwell–Yukawa black hole for plasma and also for non-plasma medium. In Sec. 5, we present our result.

2. Computation of Weak Lensing by Einstein-Nonlinear-Maxwell–Yukawa Black Hole and Gauss–Bonnet Theorem

The Einstein-nonlinear-Maxwell–Yukawa BH in a static and spherically symmetric form is given as [71]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The metric function $f(r)$ yields

$$f(r) \simeq 1 + \frac{2M}{r} - \frac{4qC_0}{r^2} + \frac{4qC_0\alpha}{3r} - qC_0\alpha^2 + \mathcal{O}(\alpha^3). \tag{2}$$

Here, M is rendered as mass of BH, C_0 is an integration constant [72], q represents charge of BH that is located at the origin and α is a positive constant and it can be chosen as $\alpha = 1$.

The optical spacetime is simply written as

$$dt^2 = \frac{dr^2}{f(r)^2} + \frac{r^2 d\phi^2}{f(r)}. \tag{3}$$

From Eq. (1), we obtain non-zero Christopher symbols as

$$\Gamma^0_{00} = \frac{-f'(r)}{f(r)}, \quad \Gamma^0_{11} = \frac{2f(r) - rf'(r)}{2rf(r)}, \quad \Gamma^1_{01} = \frac{r^2 f'(r) - f(r)2r}{2}. \tag{4}$$

Now, we calculate the Ricci Scalar that corresponds to the optical metric by using the non-zero Christopher symbols stated as follows:

$$\mathcal{R} = \frac{-f'(r)2}{2} + f''(r)f(r). \tag{5}$$

The Gaussian curvature that is computed is

$$\mathcal{K} = \frac{\text{RicciScalar}}{2}. \tag{6}$$

After simplifying, Gaussian optical curvature for Einstein-nonlinear-Maxwell–Yukawa black hole is stated as

$$\mathcal{K} = \frac{-f'(r)^2}{4} + \frac{f(r)f''(r)}{2}, \tag{7}$$

where $f(r)$ is given in Eq. (2) so that Gaussian optical curvature for Einstein-nonlinear-Maxwell–Yukawa black hole in leading order term is obtained as

$$\mathcal{K} \approx -12\frac{C_0 q}{r^4} + 4\frac{C_0 q\alpha}{3r^3} + \left(2r^{-3} - 24\frac{C_0 q}{r^5} + 4\frac{C_0 q\alpha}{r^4}\right)M. \tag{8}$$

3. Deflection Angle of Einstein-Nonlinear-Maxwell–Yukawa BHs and Gauss–Bonnet Theorem

Now, by utilizing GBT, we will calculate the deflection angle of photon by Einstein-nonlinear-Maxwell–Yukawa BH. By using GBT in the region $\mathcal{G}_R$, given as

$$\int_{\mathcal{G}_R} \mathcal{K}dS + \oint_{\partial\mathcal{G}_R} kdt + \sum_t \hat{\alpha} = 2\pi\mathcal{X}(\mathcal{G}_R), \tag{9}$$

where k represent the geodesic curvature, $\mathcal{K}$ denotes the Gaussian curvature, one can define k as $k = \bar{g}(\nabla_{\check{\alpha}}\hat{\alpha}, \check{\alpha})$ in that way $\bar{g}(\hat{\alpha}, \hat{\alpha}) = 1$, where $\hat{\alpha}$ represents the unit acceleration vector and α_t denotes the exterior angle at tth vertex, respectively. As $R \to \infty$, we obtain the jump angles approximate to $\pi/2$. Thus $\alpha_O + \alpha_S \to \pi$. Here, $\mathcal{X}(\mathcal{G}_R) = 1$ is a Euler characteristic number and $\mathcal{G}_R$ denotes the non-singular domain. Therefore, we obtain

$$\iint_{\mathcal{G}_R} \mathcal{K}dS + \oint_{\partial\mathcal{G}_R} kdt + \hat{\alpha} = 2\pi\mathcal{X}(\mathcal{G}_R), \tag{10}$$

where the total jump angle is $\hat{\alpha} = \pi$. When R approaches infinity, then the remaining part is $k(D_R) = |\nabla_{\dot{D}_R}\dot{D}_R|$. For geodesic curvature the radial component is described as

$$(\nabla_{\dot{D}_R}\dot{D}_R)^r = \dot{D}_R^\phi\partial_\phi\dot{D}_R^r + \Gamma_{\phi\phi}^r(\dot{D}_R^\phi)^2. \tag{11}$$

At R very high , $D_R := r(\phi) = R = $ const. Thus, the first term of Eq. (11) vanishes and $(\dot{D}_R^\phi)^2 = \frac{1}{f^2(r^\star)}$. Recalling $\Gamma_{\phi\phi}^r = \frac{2f(r)-rf'(r)}{2rf(r)}$, we get

$$(\nabla_{\dot{D}_R^r}\dot{D}_R^r)^r \to \frac{-1}{R}, \tag{12}$$

and $k(D_R) \to R^{-1}$, so we write $dt = Rd\phi$. Thus,

$$k(D_R)dt = d\phi. \tag{13}$$

From the previous results, we get

$$\iint_{\mathcal{G}_R} \mathcal{K}ds + \oint_{\partial\mathcal{G}_R} kdt =^{R\to\infty} \iint_{S_\infty} \mathcal{K}dS + \int_0^{\pi+\sigma} d\phi. \tag{14}$$

At 0th order, the weak field deflection limit of the light is defined as $r(t) = b/\sin\phi$. Hence, the deflection angle is given as

$$\sigma = -\int_0^\pi \int_{b/\sin\phi}^\infty \mathcal{K}\sqrt{\det\bar{g}}\,du\,d\phi, \tag{15}$$

where we put the first terms of Eq. (8) into Eq. (15), so we get the deflection angle upto leading order term as

$$\sigma \simeq \frac{4M}{b} + \frac{C_0 M \alpha q \pi}{b^2} + 8\frac{C_0 q \alpha}{3b} - 3\frac{C_0 q \pi}{b^2} - \frac{32 C_0 M q}{3b^3}. \tag{16}$$

Note that the solution (16) with $\alpha = q = 0$ reduces the deflection angle of Schwarzschild BH in the leading order terms. Moreover, it can be seen that the α parameter increases the deflection angle.

4. Weak Lensing by Einstein-Nonlinear-Maxwell–Yukawa Black Hole in a Plasma Medium

This section is based on the calculation of weak gravitational lensing by Einstein-nonlinear-Maxwell–Yukawa black hole in plasma medium. For Einstein-nonlinear-Maxwell–Yukawa black hole, the refractive index, $n(r)$, is obtained as

$$n(r) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2} f(r)}, \tag{17}$$

where electron plasma frequency is ω_e and photon frequency measured by an observer at infinity is ω_∞, then the corresponding optical metric is illustrated as

$$dt^2 = g_{ij}^{opt} dx^i dx^j = n^2(r)\left[-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2)\right]. \tag{18}$$

The metric function $f(r)$ is defined in Eq. (2) and the non-zero christopher symbols corresponding to the optical metric are computed as follows:

$$\Gamma_{00}^0 = \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2}\right)\left[\frac{-\omega_e^2 f'(r)}{2\omega_\infty^2}\right] - \frac{-f'(r)}{f(r)},$$

$$\Gamma_{01}^1 = \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2}\right)\left(\frac{-\omega_e^2 f'(r)}{2\omega_\infty^2}\right) + \frac{2f(r) - rf'(r)}{2rf(r)}, \tag{19}$$

$$\Gamma_{11}^0 = \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2}\right)\left(\frac{\omega_e^2 f(r)f'(r)r^2}{2\omega_\infty^2}\right) + \frac{r^2 f'(r) - f(r)2r}{2}.$$

Now, by using the above non-zero christopher symbols, the value of Gaussian optical curvature is found as

$$\mathcal{K} \approx 3\frac{\omega_e^2 M}{r^3 \omega_\infty^2} + 2\frac{M}{r^3} - 104\frac{C_0 q M \omega_e^2}{\omega_\infty^2 r^5} - 24\frac{C_0 q M}{r^5} - 20\frac{C_0\, q \omega_e^2}{\omega_\infty^2 r^4}$$

$$- 12\frac{C_0\, q}{r^4} + 16\frac{q\alpha C_0 \omega_e^2 M}{\omega_\infty^2 r^4} + 4\frac{q\alpha C_0 M}{r^4} + 2\frac{q\alpha\, C_0 \omega_e^2}{r^3 \omega_\infty^2} + 4/3\frac{q\alpha C_0}{r^3}. \tag{20}$$

For this, we use GBT to compute the deflection angle in order to relate it with non-plasma. As follows, for calculating angle in the weak field limits, the light beams

become a straight line. Therefore, the condition at zero order is $r = \frac{b}{\sin \phi}$. Then the GBT reduces to this simple form for calculating the deflection angle σ:

$$\sigma = - \lim_{R \to 0} \int_0^\pi \int_{\frac{b}{\sin \phi}}^R \mathcal{K} dS. \tag{21}$$

So, the deflection angle of Einstein-nonlinear-Maxwell–Yukawa BH in the presence of plasma medium is obtained as follows:

$$\sigma \simeq \frac{4M}{b} + 4\frac{q\alpha \, C_0 \, M\omega_e{}^2\pi}{b^2\omega_\infty{}^2} + \frac{C_0 \, qM\alpha \, \pi}{b^2} + 4\frac{C_0 \, q\alpha \, \omega_e{}^2}{b\omega_\infty{}^2} + 8/3\frac{qC_0 \, \alpha}{b}$$

$$- 5\frac{C_0 \, q\omega_e{}^2\pi}{b^2\omega_\infty{}^2} - 3\frac{qC_0 \, \pi}{b^2} - \frac{416 \, q \, C_0 \, M\omega_e{}^2}{9 \, b^3\omega_\infty{}^2} - \frac{32 \, C_0 \, qM}{3 \, b^3} + 6\frac{M\omega_e{}^2}{b\omega_\infty{}^2}. \tag{22}$$

The above result tells us that photon rays are moving into medium of homogeneous plasma. One can see that the plasma effect can be removed by neglecting $\frac{\omega_e}{\omega_\infty}$ term from Eq. (22) and it reduced to Eq. (16).

5. Graphical Influence of Deflection Angle on Einstein-Nonlinear-Maxwell–Yukawa Black Hole

This section of this paper comprises the graphical influence of deflection angle of Einstein-nonlinear-Maxwell–Yukawa BH. We examine the impact of different parameters on deflection angle. Here, for simplicity, we take $C_0 = c$

- **Figure 1(a) Left Plot** indicates the behavior of deflection angle with respect to b by fixing the value of M, c, α and varying q. It is to be observed that for small constant value of M and $q \geq 0$ the behavior of deflection angle gradually decreases with respect to impact parameter b. For increasing value of q, the deflection angle increases.
- **Figure 1(a) Right Plot** represents the behavior of deflection angle with respect to b by varying the mass M and taking q, c, α fixed. We noticed that for values of $M \geq 0$ the behavior of deflection angle gradually positively decreases with respect to b. But for the increasing values of M, the behavior of the deflection angle is increasing.

- **Figure 1(b) Left Plot** indicates the behavior of deflection angle with respect to b by fixing the value of α and varying $q.c$, M. It is to be observed that for decreasing the value of $\alpha < 0$, the deflection angle gradually decreases. On the other hand, the deflection angle is increasing gradually with respect to b for fixed α.
- **Figure 1(b) Right Plot** represents the behavior of deflection angle with respect to b by varying the c and taking q, α, M fixed. We noticed that for values of $b > 5$ and $c \geq 0$ the behavior of deflection angle gradually positively decreases. For the increasing value of c, the behavior of deflection angle is negatively decreasing.

(a)

(b)
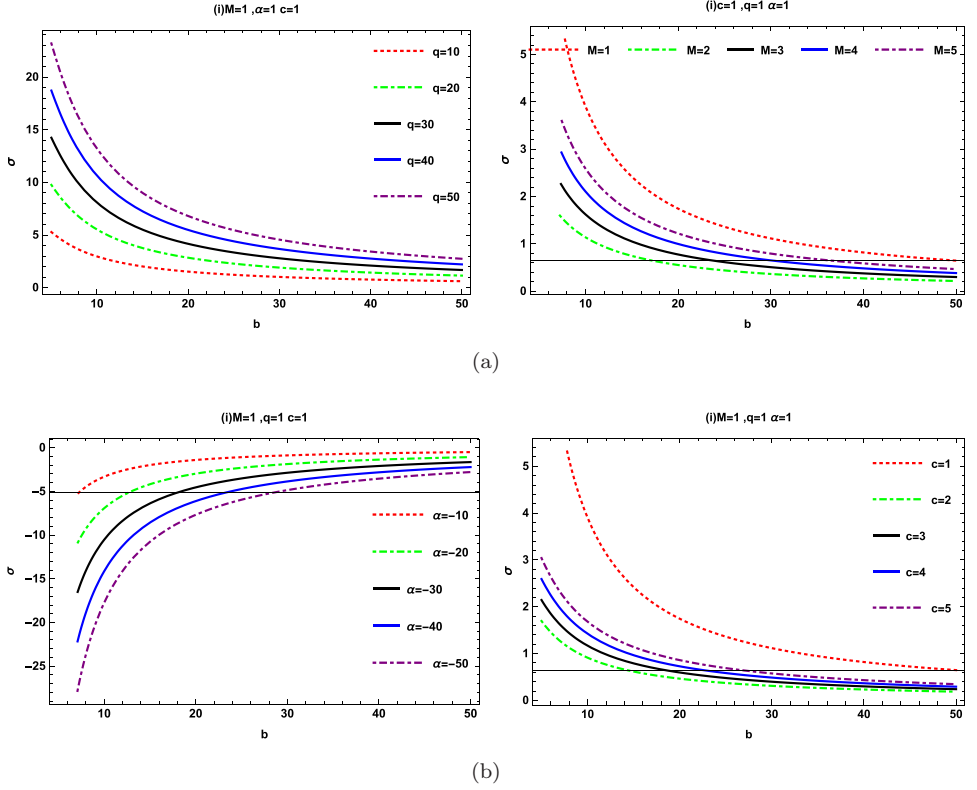
Fig. 1. (a) and (b) σ versus b.

6. Graphical Analysis for Plasma Medium

This section gives us the graphical analysis of deflection angle α in the presence of plasma medium. For simplicity, we take $C_0 = c$ in Eq. (22).

- **Figure 2(a) Left Plot** shows the behavior of deflection angle with respect to b by taking the fixed values of M, q, c, α and by varying β. The analysis shows that initially the behavior of deflection angle is positively decreasing for $\beta \geq 0$. Furthermore, for the increasing value of β, the deflection angle positively increases for fixed b.
- **Figure 2(a) Right Plot** illustrates the behavior of deflection angle with respect to b by varying BH charge q and for fixed M, α, c and β. We examined that for positive values of q, the deflection angle is gradually positively decreasing. While, for increasing values of q, the behavior of deflection angle is gradually increasing for fixed b.
- **Figure 2(b) Left Plot** examines the behavior of deflection angle with respect to b by taking the fixed values of β, q, c, M and by varying α. We noticed that for small values of α, and $\beta = 1$, $q = 1$ then the deflection angle cannot define

(a)

(b)
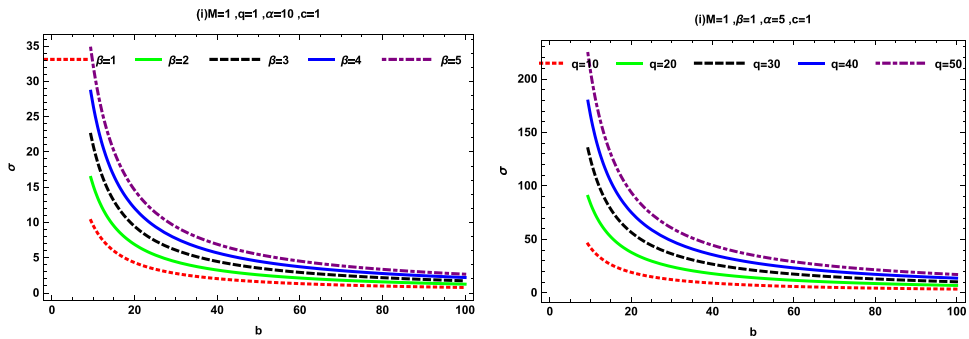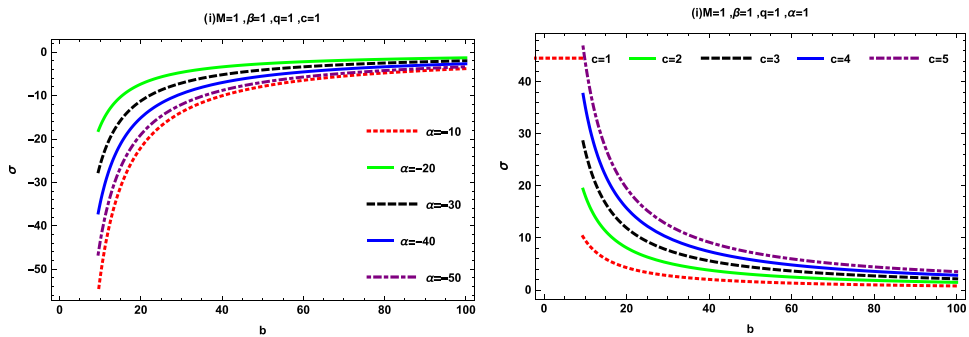
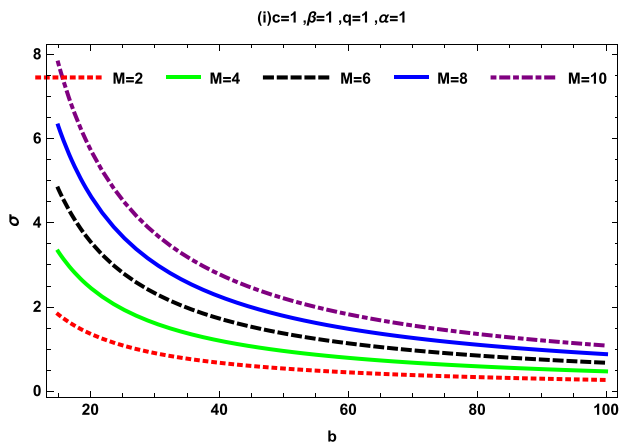Fig. 2. (a) σ versus b and (b) $\tilde{\sigma}$ versus b.

Fig. 3. σ versus b.

the behavior. While, for large values of α, the behavior of deflection angle is positively decreasing. For values of $\alpha < 0$, the behavior of deflection angle is gradually increasing.

- **Figure 2(b) Right Plot** analyzes the behavior of deflection angle with respect to b by varying c and for fixed M, α, q and β. We see that for $c \geq 0$ the deflection angle is positively decreasing. For increasing values of c, the deflection angle is increasing for fixed b.

- **Figure 3** manifests the influence of deflection angle with respect to b for constant values of q, β, c, α and varying M. It has been noted that for small range of $b < 15$, the behavior of deflection angle is not defined. On the other hand, for large $b \geq 15$ and for $M \geq 0$, the deflection angle is positively decreasing and the behavior of deflection angle is increasing for increasing values of M at fixed b.

7. Conclusion

This paper is about the investigation of deflection angle by Einstein-nonlinear-Maxwell–Yukawa BH in non-plasma medium as well as plasma medium. In this regard, we study the weak gravitational lensing by using GBT and obtain the deflection angle of photon for Einstein-nonlinear-Maxwell–Yukawa BH. The obtained deflection angle is given in Eq. (16) as follows:

$$\sigma \simeq \frac{4M}{b} + \frac{C_0 M \alpha q \pi}{b^2} + 8\frac{C_0 q \alpha}{3b} - 3\frac{C_0 q \pi}{b^2} - \frac{32 C_0 M q}{3b^3} + \mathcal{O}(q^2, M^2, C_0^2).$$

We examine that by the reduction of some parameters the obtained deflection angle converted into the Schwarzschild deflection angle up to the first-order terms. We also discuss the graphical effect of different parameters on deflection angle by Einstein-nonlinear-Maxwell–Yukawa BH. We also observed that the deflection angle in the presence of plasma medium given by Eq. (22) is

$$\sigma \simeq \frac{4M}{b} + 4\frac{q\alpha C_0 M \omega_e^2 \pi}{b^2 \omega_\infty^2} + \frac{C_0 q M \alpha \pi}{b^2} + 4\frac{C_0 q \alpha \omega_e^2}{b \omega_\infty^2} + 8/3\frac{q C_0 \alpha}{b}$$

$$- 5\frac{C_0 q \omega_e^2 \pi}{b^2 \omega_\infty^2} - 3\frac{q C_0 \pi}{b^2} - \frac{416 q C_0 M \omega_e^2}{9 b^3 \omega_\infty^2} - \frac{32 C_0 q M}{3b^3}$$

$$+ 6\frac{M \omega_e^2}{b \omega_\infty^2} + \mathcal{O}(q^2, M^2, C_0^2, \omega_e^3). \tag{23}$$

When $\frac{\omega_e}{\omega_\infty}$ approaches zero, the plasma effect is removed. Furthermore, we scrutinized the graphical impact of deflection angle on Einstein-nonlinear-Maxwell–Yukawa BH in plasma medium versus some parameters. One can conclude that the deflection of photon is found outside the lensing area which points that the gravitational lensing effect has a global and even topological effect. Hence, it is our belief that studying on the Virbhadra–Ellis lens equation can help us improve

the computation of the image positions, Einstein ring radii, magnification factors and the magnification ratio. This is going to be our next problem in the near future. Moreover, it will be interesting to investigate the deflection angle of BHs in MAXWELL $f(R)$ gravity theories and in fourth-order gravity theories using the GBT in future [73–75].

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